SESSION THREE
AREA MEASUREMENT AND FORMULAS

Outcomes

- Understand the concept of area of a figure
- Be able to find the area of a rectangle and understand the formula base times height
- Be able to find the area of a right triangle and understand the formula base times height divided by two
- Be able to find the area of an arbitrary triangle and understand the formula
- Be able to find the area of a parallelogram and understand the formula base times height
- Derive the formula for the area of the trapezoid

Overview

The concept of area is a basic one in mathematics. However, often participants do not know why the formulas for areas of shapes are valid. The activities in this session will help participants see how the formulas for the areas for different shapes are related to each other.

Time

15-20 minutes	Definitions of area and perimeter
10–15 minutes	Area of rectangle
10-15 minutes	Area of right triangle
25-35 minutes	Area of other triangles
15–20 minutes	Area of prallelogram
15–20 minutes	Area of trapezoid
5 minutes	Relations among formulas

Materials

Facilitator	Transparencies (Eng. & Spanish)
• Geoboard for overhead (11 pegs per side) and rubberbands	*BLM 21: Area and Perimeter* *BLM 25: A Puzzle for the Area of a Parallelogram*
Participant	**Handouts (English & Spanish)**
• Geoboard (11 pegs per side) • Rubberbands (different colors) • Calculators • Rulers (optional if geoboards are not available) • Scissors • Poster paper • Cardboard and glue, if cardstock is not used for BLM 25 and BLM 26	**One per participant for class** *BLM 21: Area and Perimeter* *BLMs 23.1-5: Area on the Geoboard* *BLMs 24.1-2: Hints and Solutions* *BLM 25: A Puzzle for the Area of a Parallelogram* *BLM 26: The Area of a Trapezoid* **One per participant for home** *BLM 22: Grid Paper*

Activities

Preparation of Classroom	Notes
1. Prepare handouts for distribution for class activities. 2. Place the name cards from last class near the front of the room where participants can easily find them. 3. Have participant materials, manipulatives, and handouts on the tables.	Handouts can either be placed on the tables before the session begins or passed out at the beginning of each activity. Cardstock can be used in place of cardboard whenever cardboard is referenced in the sessions' activities.
Area and Perimeter (15-20 minutes)	
Materials and handouts: • Geoboard and rubberbands • BLM 21: Area and Perimeter **Opening Activity** 1. The goal of the first activity is to develop the concept that area is the number of unit squares contained in the figure. Participants need to realize that no formulas are needed for the concept. Formulas are just convenient ways to compute the area of some particular figures. A common misconception is that the concept of area is the formula for area. Often participants, when asked what area is, they respond base times height or another formula. 2. Ask participants to find the area and the perimeter of the shapes in the first handout. Explain that the grid used is cm^2. Ask participants to describe the method they used to find the area of the different shapes in order to come up with a definition of area in their small group. Have different groups share their definitions, and write them so all can see them. After some discussion, participants may see what is common to the definitions of the different groups and will arrive at a more precise definition such as "The area is the number of unit squares enclosed in the figure." 3. In the same way, participants can find the perimeter of the figures, and try to reach a definition such as The perimeter is the length of the boundary of the figure, that is, the number of unit segments around a figure. 4. Ask participants to describe in their own words what area means and what perimeter means. Ask them to give examples of when area is used, and when perimeter is used. 5. A way to find the area of a figure is to count the number of unit squares contained in it. In some cases that is the easiest way to find the area. Later, participants will see that for special figures, like rectangles, there are efficient ways to count the number of unit squares	

Activities

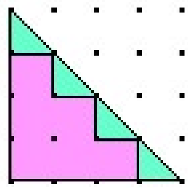
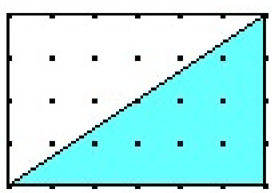
Area and Perimeter (continued)	
contained. However, participants need to realize that a formula such as base times height is not the definition of area, it is just a shortcut to find the area of particular shapes such as rectangles.	

Area on the Geoboard (1 hour and 25 minutes)	Notes
Materials and handouts: • Geoboard and rubberbands, calculators, and scissors • BLM 22: Grid Paper & scissors • BLM 23.1-5: Area on the Geoboard • BLM 25: A Puzzle for the Area of a Parallelogram	
1. The geoboard is a board that has pegs forming a square grid. They are commercially available (about $6). Participants can form geometric shapes using rubber bands.	
3. For these activities the square that is between four pegs on the geoboard will be considered of area 1 unit square. It is important that participants count intervals between pegs, rather than pegs to describe a figure. Thus the unit square is a one by one square.	
4. It is important to stress that the basic unit of length is the distance between two contiguous pegs either horizontally or vertically. A common mistake done by participants, as well as children, is to count as a unit the distance between pegs that are "diagonally" placed with respect to each other. The instructor may point that the distance along the diagonal of a square is bigger than the distance along a side.	
5. The use of the geoboard is highly recommended. Participants can build shapes very quickly. They can also add auxiliary lines and "erase" them without a problem. This contributes to a risk free environment where participants can explore their own methods and strategies to compute areas of figures.	If participants do not have a geoboard, they can use the grid (fournd on BLM 22) and a ruler to trace the figures.
The area of a rectangle (10-15 minutes) In the case of a rectangle, it is important to realize that to find its area we are still finding the number of unit squares. When we use the formula base times height, it is a way to find out how many squares there are in a row and then multiply this number by the number of rows. Participants often have not thought through the meaning of this very familiar formula. Often they mistake familiarity with understanding.	It is very likely that some of the participants will remember that the area of a rectangle is base times height, and some will even remember that the area of a triangle is given by one half base times height. However, it is also very likely that they do not know why the formula works.

Activities

Area on the Geoboard (continued)	Notes
Activity 1 Ask participants to construct one unit square on the geoboard with a rubber band. Ask them to construct a rectangle that contains the unit square. How many unit squares can they fit into their rectangles? **Activity 2** 1. Ask participants to construct a rectangle on the geoboard that has a base of five units and a height of four units. Let them count the number of unit squares. 2. Let participants divide the rectangle into rows. The rectangle has four rows. There are five squares in each row. Besides counting the squares one by one, they can use another method to find the total number of squares in the rectangle. Because there are five squares per row, and there are four rows, the total number of squares is 5 + 5 + 5 + 5 = 4 x 5 = 20. Participants should notice that the number of squares per row is given by the length of the base of the rectangle, and that the number of rows is given by its height. That is why the formula base × height is used to compute the area of a rectangle. 3. Participants can also divide the five by four rectangle into columns. It has five columns. There are four squares in each column. The total number of squares in the rectangle can also be computed as 4 + 4 + 4 + 4 + 4 = 5 × 4. In this case the number of columns is given by the length of the base, and the number of squares in each column is given by the height of the rectangle. 4. Ask participants to explain in their own words how the methods of counting squares in columns or rows relate to the usual formula to compute the area of a rectangle. 5. From the basic formula of the rectangle the other formulas are derived. An important aspect is that participants see that the area formulas are related to each other and that there is no need to memorize all the formulas. Once the connections between them are seen, the particular formulas can be derived when needed.	Having a calculator available may help some of the participants focus on the angles rather than on the computations.
Activity 3. The area of a right triangle 1. Ask participants to construct a right triangle with one of the sides forming the right angle parallel to the border of the geoboard. Let participants choose their own right triangle. For many participants the right triangle with 45°	Participants are more confident when they are allowed to use their own shapes. In general, there will be a variety of different shapes of each kind to allow comparisons.

Activities

Area on the Geoboard (continued)	Notes
Activity 3 (continued) angles is the easiest triangle to determine its area. They can count half squares and whole squares to find the total number of squares contained in the triangle. For example, a right triangle with a base of four units and height of four units has six whole squares and four half squares. The total area is eight unit squares. 	Sharing alternative methods is very instructive for every body. Participants can hold their geoboards up so that everybody can see what shape they are using. Use of colored rubber bands helps see what segments are referred to.
2. It is important that participants also see other strategies, such as finding a square with twice the area. The strategy of enclosing the right triangle in a rectangle can be used with other triangles where fractions of squares are not so easy to determine. 3. It is also important to have other kinds of right triangles that are not isosceles. In that case the figure that encloses the triangle will be a rectangle that is not a square. 4. Ask participants to construct a right triangle with one of the sides forming the right angle parallel to the border that is not isosceles.  Let them know that the right triangle is half of a rectangle. They can use this relation to find the area of the triangle. The total area of the rectangle shown above is 6 × 4 = 24. Because the area of the right triangle is half of the area of the rectangle, its area is $\frac{6 \times 4}{2}$, that is base × height divided by two. In general, if *b* is the base of the triangle, and *h* is its height we can write the formula for the area of a right triangle as $\frac{b \times h}{2}$.	The instructor may use a transparent geoboard for the overhead projector. By sharing, not only do participants learn from each other, but also they realize that there are many ways to solve a given problem. This in turn can help participants listen to their children's method rather than just telling them how to solve a problem in a prescribed way.

Activities

Area on the Geoboard (continued)	Notes

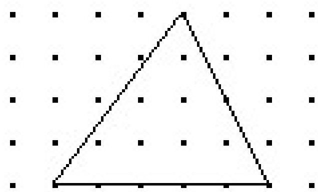
Activity 3 (continued)

5. The main point of this activity is to see that the area of the right triangle is half the area of the corresponding rectangle. For many participants this is the first time they see where the divided by two of the formula base times height divided by two comes from.

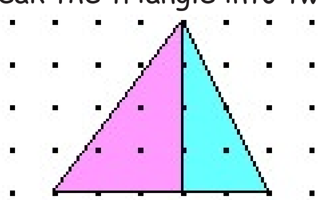
6. The right triangle will be the leading particular case to obtain the formula for all other kinds of triangles.

Activity 4. Areas of Other Triangles (25-35 min.)

1. Ask participants to construct a triangle that has only one side parallel to one of the borders of the geoboard, and so that the angles at the base are less than 90°.



2. Participants can use what they learned before to compute the area of a right triangle. They can use an extra rubber band to break the triangle into two right triangles.

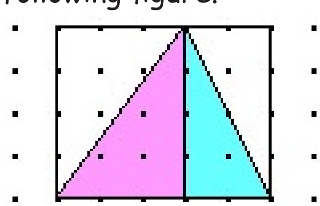


The purple triangle has a base of 3 units and a height of 4; its area is $(3 \times 4)/2 = 6$. The blue triangle has an area of $(2 \times 4) / 2 = 4$. The total area of the triangle is therefore $6 + 4 = 10$. We can also express this as

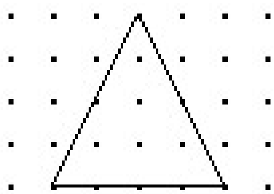
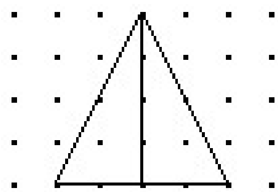
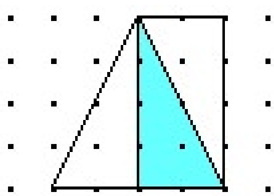
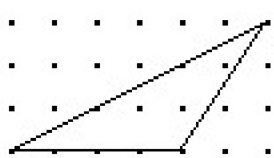
$$\frac{3\times 4}{2}+\frac{2\times 4}{2}=\frac{4(3+2)}{2}=\frac{4\times 5}{2}.$$

Notice that 5 is the base of the triangle and 4 is its height.

3. A second method frequently used by participants is illustrated by the following figure.



Activities

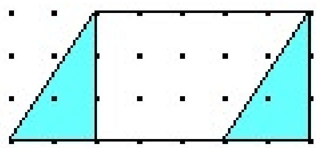
Area on the Geoboard (continued)	Notes
Participants can build a rectangle around the triangle with the same base and the same height. The area of the triangle will be half the area of the rectangle. Participants can see this by dividing the original triangle with its height. The outer rectangle will be divided into two rectangles. The purple triangle is half the area of the rectangle on the left. The blue triangle is half the rectangle on the right. Because the area of the rectangle is 5 × 4, the area of the triangle is (5 × 4)/2. 4. When participants have an isosceles triangle (one with two legs congruent), they can show that its area is the same as the area of a rectangle with half the base and the same height as the original triangle. The area of the rectangle is h × b/2, which is the same as (b × h)/2.    **The case of the obtuse triangle** A case that is quite difficult for many participants if that of a triangle with an obtuse angle with one of the smaller sides parallel to the border of the geoboard.  Participants will deal with this case in Activity 6. **Activity 5. The area of a parallelogram (15-20 min)** 1. Ask participants to construct a parallelogram that has one base parallel to the border of the geoboard. It is important that participants choose their own parallelogram. Ask them to find the area of their parallelogram.	Often, even though participants are able to construct a rectangle around a triangle with the same base and the same height, they find it hard to give a reason why the area of the triangle is one half the area of the rectangle. The instructor may help by placing a rubber band along the height of the triangle to break it into two right triangles. Often this is all participants need to see that each of the two right triangles is half of the corresponding rectangles. Wait until the end of the lesson to handout the Hints and Solutions to activities 5 and 6.

Activities

Area on the Geoboard (continued)

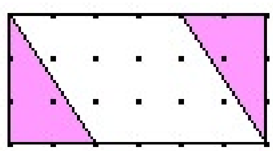
There are different ways to obtain the area of a parallelogram. One way is to count directly the number of unit squares contained in it. This method is easy to use if the sides are inclined at a 45° angle.

There are other two commonly used methods. The first is to "cut" one triangle from one end of the parallelogram and "paste" it on the other end to form a rectangle with the same base and the same height. In this case the relation to the formula of the area of the parallelogram as base times height is immediate.



One important point that participants need to make explicit is that the height of the parallelogram is used, not its side.

2. Another method is to draw a rectangle with the same height that encompasses the parallelogram.



With this method, participants usually compute the area of the parallelogram by subtraction. The instructor may need to guide participants to see the connection of this subtraction method with the formula.

This method works also for parallelograms where the base is small and the height big, so that it is not possible to use the method to cut a triangle on one side and paste it on the other. To illustrate, here we have a parallelogram that has a base of two units and a height of seven units. Participants can form a rectangle around the parallelogram. Its base is 5 units and its height 7, so its area is 5 × 7. The area of the two purple right triangles is 3 × 7. The difference is 5 × 7 - 3 × 7 = (5 - 3) × 7 = 2 × 7. That is, the base of the parallelogram times its height.

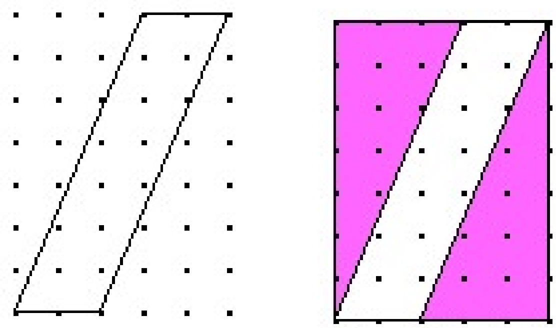
Notes

This is a fun activitiy and adds variety from using geoboards. Participants will need the grid paper on BLM 22 and scissors.

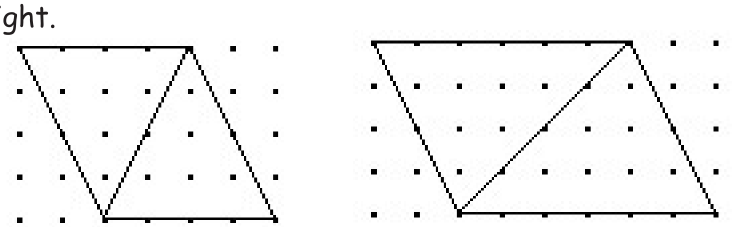
Activities

Area on the Geoboard (continued)	Notes

Activity 5 (continued)



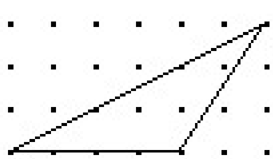
3. Sometimes participants use an alternative way to find the area of a parallelogram. Parallelograms can be divided into two congruent triangles by constructing one of the diagonals. Let b be the base of the parallelogram, h its height.



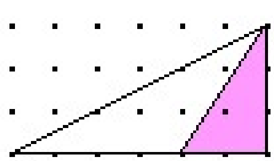
The area of each triangle is 1/2 x b x h. The area of the parallelogram is therefore 2 x 1/2 x b x h, that is b x h.

Activity 6. The Obtuse Triangle

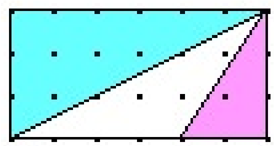
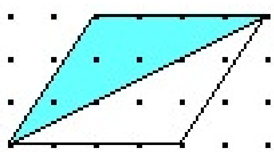
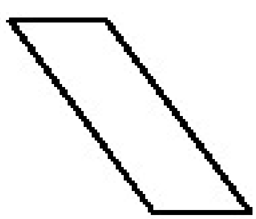
1. A triangle with an obtuse angle with one of the smaller sides parallel to the border of the geoboard will require a different approach.



The instructor may guide participants by pointing to some right triangles that could be used in connection with this case.



Activities

Area on the Geoboard (continued)	Notes
Activity 6 (continued) 2. Participants usually find the area of the desired triangle by subtraction. They will need some guidance to see how their method is related to the formula *base times height divided by two*.  To illustrate, here we have a triangle with a base of 4, and height of 3, with an obtuse angle.	Computing areas in different ways offers the opportunity to provide meaning to algebraic manipulations of equation and expressions. The instructor will guide participants by providing explicit notation for the parts of the different figures. By connecting the symbolic transformation to different interpretations participants will have ways to see why the expressions do indeed represent the same quantity.
3. One way to find the area is by constructing a small right triangle, so that the original triangle together with this form a right triangle. The area of the original triangle will be the difference of the areas of the two right triangles. The area of big right triangle is *1/2(6x3)*. The area of the small right triangle is *1/2(2x3)*. The difference is *1/2(6x3). - 1/2(2x3) = 1/3x3x(6-2) = 1/2x3x4*. Participants should notice that again we have that the area is one half the base times the height.	However, many participants are not fluent with algebraic manipulations and it is easy for them to get lost. Concrete examples with numbers are easier for them to follow.
4. Another approach is to relate this kind of triangle to the area of the parallelogram. The triangle is half of a parallelogram with the same base and same height. 	
Additional Activity. A Puzzle for the Area of a Parallelogram 1. A case of parallelogram that is especially difficult is when the base is small and the height is big, and it is not possible to cut one right triangle from one side to paste it on the other to form a rectangle. 	Pieces of cardboard or cardstock can also be very effective to show the relation between the different shapes and show the relationship between the corresponding formulas for area.

Activities

Area on the Geoboard (continued)	Notes

Additional Activity (continued)
This case warrants a different approach in addition to the geoboard. Participants will need to have the rectangle and parallelogram shown in the handouts pasted on cardboard and cut out in advance for this activity.

2. Ask participants to compare the parallelogram and the rectangle. They need to realize that they have the same base (short side), and the same height.

3. Ask participants to use the parallelogram and the triangle from the puzzle in the handout.

4. Now ask them to use the rectangle and the triangle to form the puzzle. Ask them what they can say about the area of the parallelogram and the area of the rectangle.

5. The instructor may need to guide participants to see the connection of the area obtained with the puzzle and the formula.

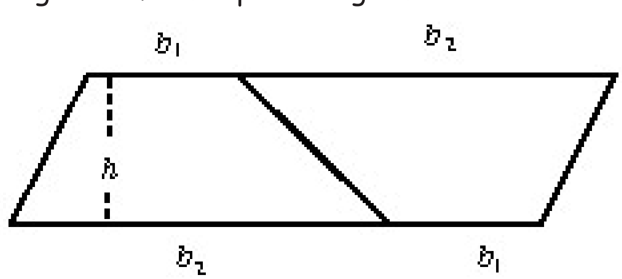
Connections (15-20 minutes)	

Handout:

- BLM 26: The Area of a Trapezoid

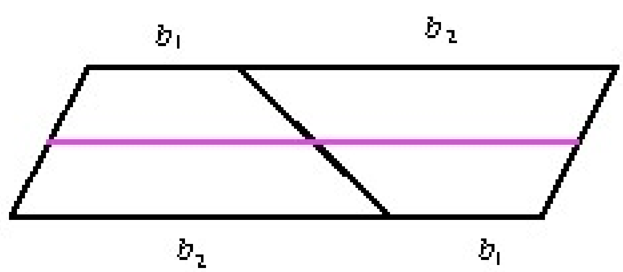
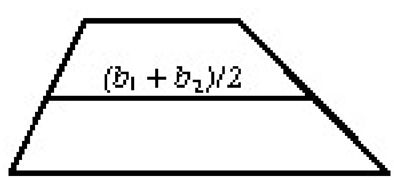
The Area of a Trapezoid
1. Two congruent cardboard trapezoids are quite convenient for developing the formula for the area of the trapezoid. Participants will need to paste on cardboard and cut the two trapezoids in advance for this activity.

2. Ask participants to turn one of the trapezoids so that both together form a parallelogram.

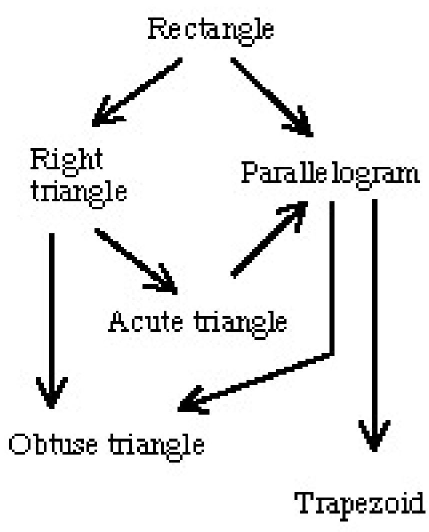


3. Ask participants to describe the length of the base of the parallelogram in terms of the bases of the trapezoids *($b_1 + b_2$)*. Ask participants to compare the height of the parallelogram with the height of the trapezoids (it is the same *h*).

Activities

Connections (continued)	Notes
4. Ask participants to describe the area of the parallelogram with a formula. The area is $h(b_1 + b_2)$. Ask them to describe the relation between the area of one of the trapezoids and the area of the parallelogram. The area of the trapezoid is half the area of the parallelogram. Ask them to give a formula for the area of the trapezoid $A = h(b_1+b_2)/2$. **Another interpretation for the formula.** 1. Ask participants to find the midpoint of the non parallel sides of each of the trapezoid and to draw a parallel to the bases through these points. This line is called the median of the trapezoid. Ask participants to turn one of the trapezoids to form again a parallelogram. Participants will see that twice the median is equal to the sum of the bases of the trapezoid, $b_1 + b_2$.   Participants will see that the length of the line parallel to the two bases of the trapezoid that is equidistant from them has a length of $(b_1+b_2)/2$. So another interpretation of the formula for the area of the trapezoid is to multiply the length of this median times the height. 2. The instructor may want to guide participants see how the formulas for the area of the parallelogram is a special case of the formula for the trapezoid.	

Activities

Closure	
Materials: • Chart paper and color markers **A diagram for the relations among area formulas.** The formulas for the area of the rectangle, the right triangle, triangles in general, as well as the formula for the area of the parallelogram, and the trapezoid are related. Ask participants to reflect back on the session and describe in their own words how the formulas are related. It is important that participants realize that they can deduce a formula from other basic formulas, using fundamental properties of area. The formula for the area of the rectangle is the most basic one. From it the other formulas can be deduced. Also, participants can realize that there are different routes they can follow to obtain the formulas. The diagram illustrates some of the connections.  	Have participants work in small groups and then process as a whole group.
Take Home Activities	
Have participants take home any activities that are not completed in class.	
Preparation for the Next Session	
Collect name cards for use in the next sessions.	