SESSION SIX
PYTHAGOREAN THEOREM

Outcomes

- Establish the Pythagorean relationship among the squares of the sides of a right triangle using a puzzle and area properties of squares and parallelograms
- Give an alternative proof of the Pythagorean Theorem using squares and four right triangles
- Apply the Pythagorean Theorem to determine areas of tilted squares on the geoboard
- Determine the distance between any two points on the geoboard
- Find examples that show the inverse of the Pythagorean Theorem, this, if the triangle does not have a right angle then $a^2 + b^2 \neq c^2$

Overview

The main purpose of this session is to let participants become acquainted with one of the most important theorems of geometry. Proving the theorem using a concrete approach will be beneficial even for participants who remember the result.

Time

30-35 minutes	Participants solve a puzzle in multiple ways to establish the Pythagorean Theorem.
10–15 minutes	Participants use an alternative approach to prove the theorem using four right tiangles and the squares of the sides.
20-25 minutes	Apply the Pythagorean Theorem on the geoboard to compute the areas of tilted squares.
5 minutes	Compute the length of the side of a tilted square.
10 minutes	Compute the distance between arbitrary points on the geoboard.
10 minutes	Explore the inverse of the Pythagorean Theorem.

Materials

Facilitator	Transparencies (Eng. & Spanish)
• Geoboard for the overhead projector **Optional materials if facilitator prepares puzzle pieces before class:** • Cardboard (optional) • Glue (optional)	*BLM 48: Pythagorean Puzzle Pieces Set A* *BLM 49: Pythagorean Puzzle A* *BLM 51: Pythagorean Puzzle Pieces Set B* *BLM 52: Pythagorean Puzzle B - Activities*
Participant	**Handouts (English & Spanish)**
• Geoboard and rubber bands • Calculators **Optional materials if participants are to cut puzzle pieces during activity:** • Scissors	**One per participant for class (copy on cardstock if participants are to cut out)** *BLM 48: Pythagorean Puzzle Pieces Set A* *BLM 51: Pythagorean Puzzle Pieces Set B* **One per participant for class** (single sided) *BLM 49: Pythagorean Puzzle A* *BLM 50: Pythagorean Puzzle A - Activities* *BLM 52: Pythagorean Puzzle B - Activities* **One per participant for class** *BLM 53.1-2: Pythagoras on the Geoboard* *BLM 54: Inverse of Pythagorean Theorem*

Activities

<table>
<tr><th>Preparation of Classroom</th><th>Notes</th></tr>
<tr><td>1. Place the name cards from last class near the front of the room where participants can easily find them.

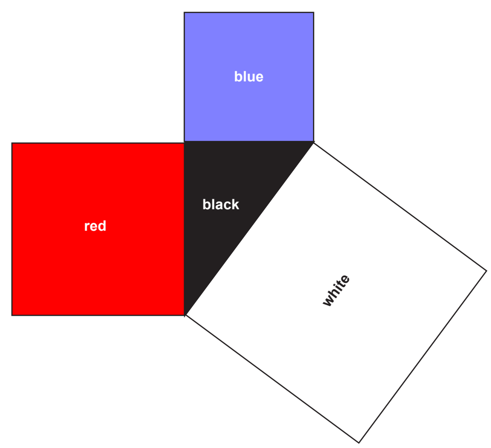
2. Have participant materials and handouts on the tables.

3. Cut out the six pieces on BLM 48: Pythagorean Puzzle Pieces Set A and the seven pieces on BLM 51: Pythagorean Puzzle Pieces Set B and paste on cardboard or copy on cardstock and have particpants cut out.</td><td>Handouts can either be placed on the tables before the session begins or passed out at the beginning of each activity.</td></tr>
<tr><th>Pythagorean Puzzle (30-35 minutes)</th><th></th></tr>
<tr><td>1. Materials and handouts:
• Six Pythagorean puzzle pieces from set A
• BLM 49: Pythagorean Puzzle A
• BLM 50: Pythagorean Puzzle A - Activities

2. Give the six pieces to participants before you give them the Pythagorean Puzzle A. Participants should take a good look at the different pieces, compare them, and describe them in their own words. They should realize that the triangle is a right triangle and they should identify the right angle. You may remind them that the two sides of the triangle forming the right angle are called legs, and the longest side, the one that is opposite to the right angle is called the hypotenuse. They should also notice that for each side of the triangle there is a square whose side is exactly the same length. The side of the white square is equal to the hypotenuse, the side of the blue square is equal to one of the legs, and the side of the red square is equal to the other leg.

Activity 1
At this point it is convenient for them to have the handout with the outline of the puzzle. Ask participants to fill the puzzle using the three squares and the right triangle. Participants realize that by using the three squares and the right triangle they can fill completely the inside of the outline. With the figure formed, participants should see and be able to describe the relation of the sides of the squares to the corresponding sides of the right triangle.

Activity 2
Ask participants to take out the red square and use instead the red parallelogram and the other pieces to fill the puzzle. Most participants do not have major problems forming this puzzle using the red parallelogram, the blue square, the white square, and the right triangle.</td><td>In general, participants do not have a problem forming the puzzle using the three squares and the right triangle.



The puzzle formed

Some participants may need some help. Rather than solving the puzzle for them, the instructor may give a hint such as asking them to place the two squares in their corresponding places first, and then fit the triangle and the parallelogram in the remaining space. Occasionally the instructor will need to point to a participant the he or she needs to flip the red parallelogram so that it fits in the remaining space.</td></tr>
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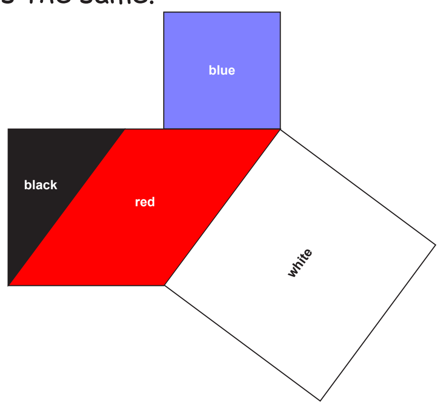
Activities

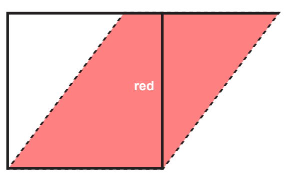
Pythagorean Puzzle (continued)

Activity 2 (continued)

a) The red parallelogram and the red square have the same area because the other three pieces remain the same and the total area is the same.

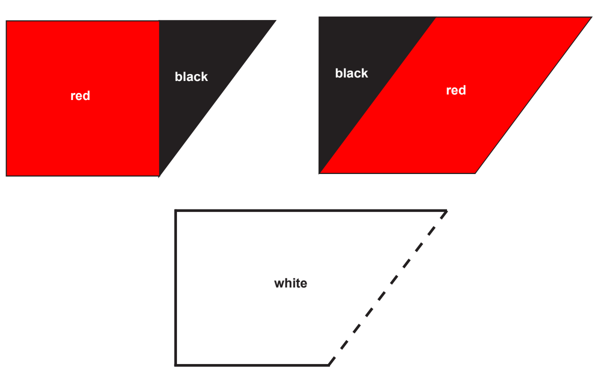


b) Participants can verify that the base of the red square is equal to one base of the red parallelogram. They can compare the height of the red square with the height of the red parallelogram.



For both the square and the parallelogram, the formula to compute the area is base times height. Because the square and the parallelogram have equal bases and equal height, they also have the same area.

c) Another way to see that the red parallelogram and the red square have the same area is by noticing that the red square plus the right triangle cover the same area as the red parallelogram plus the same triangle.



Notes

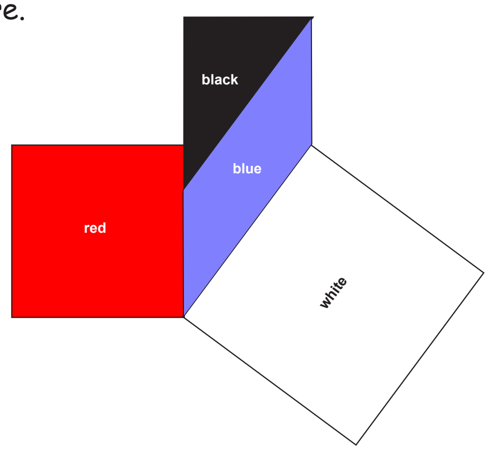
It is very important that participants realize that the red square and the red parallelogram have the same area. Encourage participants to justify this fact in their own way. Remind participants what pieces were used the first time the puzzle was formed. Different arguments can be given. Here we give three of the most common arguments. **See a-c with illustrations.**

Activities

Pythagorean Puzzle (continued)	Notes

Activity 3

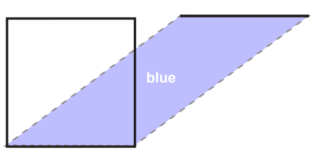
1. Ask participants to form the puzzle using the blue parallelogram, the red square, the right triangle, and the white square.



2. Ask them to compare this way of solving the puzzle with the first one. What can they say about the area of the blue parallelogram and the area of the blue square they did not use?

Again, participants should realize that the blue square and the blue parallelogram have the same area, and be able to justify this in their own way.

3. They can verify that the base of the blue square is equal to one side of the blue parallelogram, and compare the height of the blue square with the height of the blue parallelogram.



Again, the formula for the area of the parallelogram is base times height, same as for the square. Because they have the same base and the same height, their areas are equal.

Activity 4

1. Ask participants to form the puzzle using all the pieces except the white square.

Notes: Some participants may have a little more trouble forming the puzzle with the blue parallelogram, the red square, the white square, and the right triangle. One way to help is to point where the right triangle should be placed.

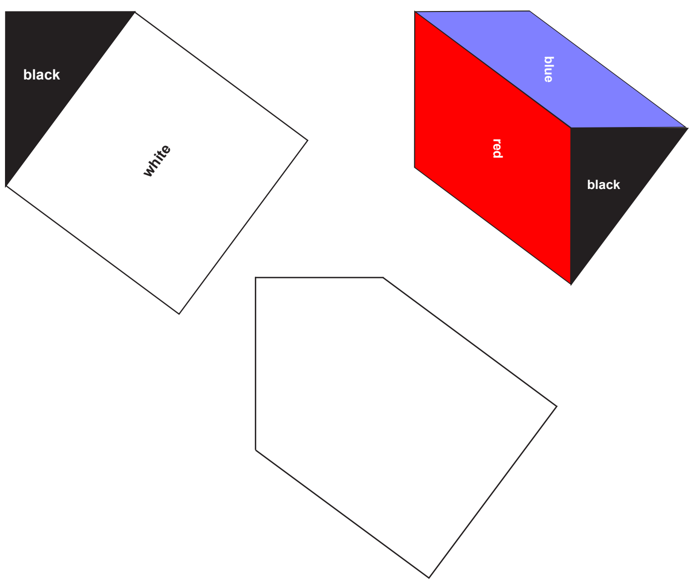
Activities

Pythagorean Puzzle (continued)

Activity 4 (continued)

Participants should realize that the area of the two parallelograms together is the same as the area of the white square, and be able to justify this fact. A common argument is that of substitution. Compared to the pieces used in the first activity, the two parallelograms substitute the white square. The total area is the same and all other pieces are the same; therefore the two parallelograms together have the same area as the white square.

2. Another way to see that the two parallelograms together have the same area as the white square is by using the right triangle. Participants will notice that the white square plus the right triangle have the same area as the two parallelograms plus the right triangle.



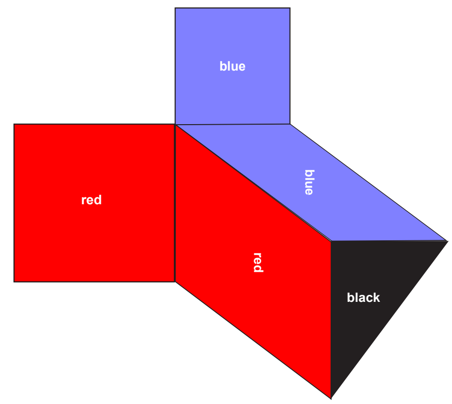
Activity 5

1. In this part, ask participants to work in small groups to tie together all the previous discoveries to determine what is the relation of the areas of the squares on the legs on one hand and the area of the square on the hypotenuse on the other. They will need to use deductive thinking to show that the sum of the areas of the squares on the legs is equal to the area of the square on the hypotenuse.

2. It will be helpful for groups to recapitulate what they have ascertained so far. They saw that the red parallelogram has the same area as the red square, that the blue parallelogram has the same area as the blue square,

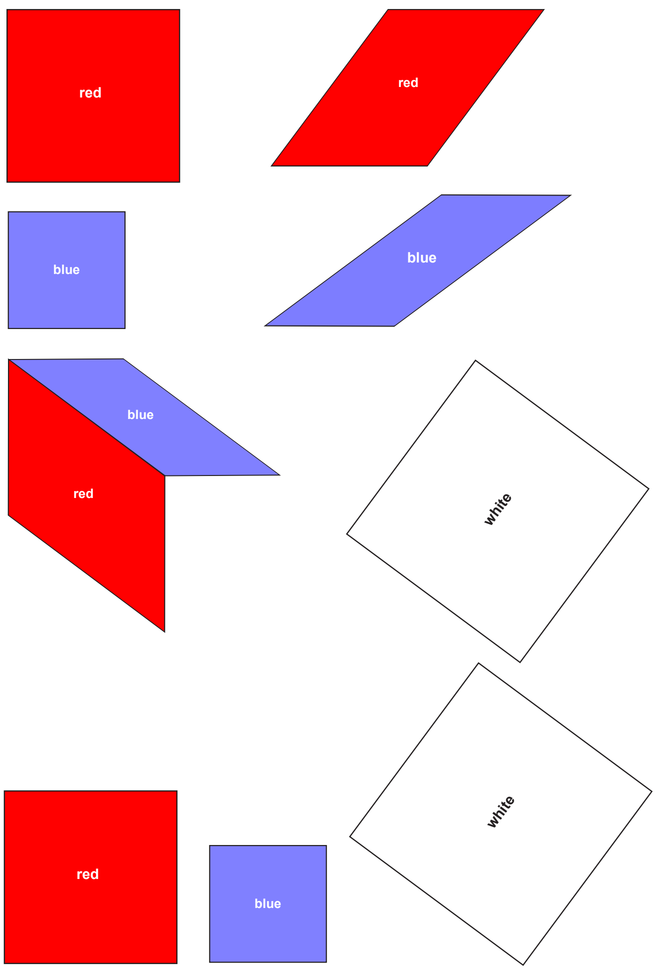
Notes

By now, most participants do not have any problem forming the puzzle using the two parallelograms, the red square, the blue square, and the right triangle. If needed, point where the right triangle is placed.



Activities

Pythagorean Puzzle (continued)	Notes
Activity 5 (continued) and that the area of the two parallelograms together was equal to the area of the white square.	



Most participants will be able to make the necessary deductions or be able to follow the deductive thinking, and conclude that the area of the blue square plus the area of the red square is equal to the area of the white square.

3. At this point some participants will recognize the Pythagorean theorem. For others, it will be necessary to label the sides of the triangle. Label the hypotenuse as c, and the legs as a and b. Then the theorem can be expressed as saying that in a right triangle with legs a and b, and hypotenuse c, $a^2 + b^2 = c^2$. For some participants, these activities will constitute their first actual proof of this result that they learned as a rule. For other participants this will be a new result.

Notes: Some participants may not be ready for geometrical thinking at this level and they will try to find an empirical proof that the sum of the areas of the two squares on the legs of the right triangle is equal to the area of the square on the hypotenuse by overlapping the squares, and by making imaginary cutting and pasting.

Activities

Another Proof of the Theorem Using Four Right Triangles (10-15 minutes)

Materials and handouts:

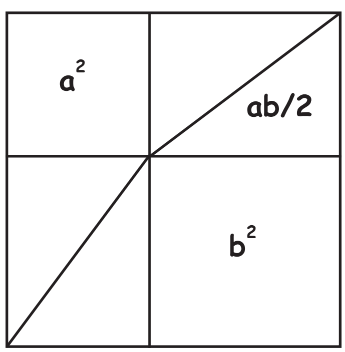
- Seven Pythagorean puzzle pieces from set B
- BLM 52: Pythagorean Puzzle B - Activities

Activity 1

1. Hand out the seven puzzle pieces of four congruent right triangles and three squares corresponding to the sides of one of the triangles. Blue and red squares will correspond to the legs of the triangle and a white square to the hypotenuse. Ask participants to fill the square frame in the handout using the four triangles and the red and blue squares.

2. Ask them to describe the total area in terms of the areas of the pieces. You may want to introduce suitable notation to facilitate the discussion of the areas.

3. If the shorter leg of the triangle is a, b the other leg, and c the hypotenuse, then the area of the squares used is a, and b, and the areas of the triangles is $ab/2$. The total area of the square frame is therefore $a^2 + b^2 + 4\,ab/2$, that is, $a^2 + b^2 + 2\,ab$.



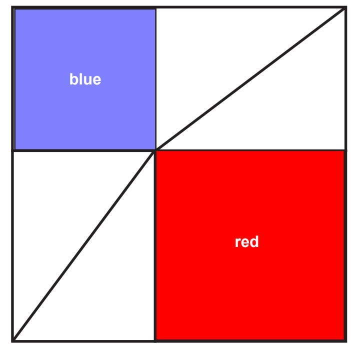
You may want to point out that the side of the square frame is $a + b$, so that its area is $(a + b)^2$. Thus we have a geometrical representation of the algebraic identity $(a + b)^2 = a^2 + b^2 + 2\,ab$.

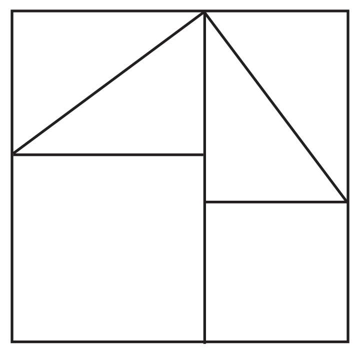
Activity 2

1. Ask participants to fill the square frame using the same four right triangles and the white square. (see next page)

Notes

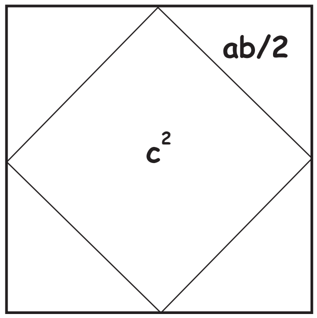
Usually participants do not have a problem filling the square frame with the given pieces. Different solutions are possible.





This is a good opportunity to make a connection to algebra.

Activities

Another Proof of the Theorem Using Four Right Triangles (continued)	Notes
Activity 2 (continued)  2. Ask participants to determine the total area as sum of the pieces. The total area is $c^2 + 4ab/2$ 3. Ask what they can say about the area of the white square and the blue and red squares together. Most participants will conclude that the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the legs of the right triangle.	
Connections: Pythagoras on the Geoboard (20-25 minutes)	
Materials and handouts: • Geoboards & bands • BLMs 53.1-2: Pythagoras on the Geoboard • BLM 54: Inverse of Pythagorean Theorem **Activity 1 (5 minutes)** 1. Ask participants to build a tilted square on the geoboard. 2. Ask them to construct a right triangle so that one side of the tilted square is the hypotenuse of the right triangle, and the legs of the triangle are parallel to the borders of the geoboard, as illustrated in figure 3. 3. Ask them to construct squares on the legs of the right angle. 4. Ask them to find the areas of the squares on the legs and add them to obtain the area of the square on the hypotenuse. 5. Ask them to find the area of the square on the hypotenuse of the given figures.	Some participants may find the area of the tilted square without using the Pythagorean theorem, by inscribing the given square into a bigger square with sides parallel to the border of the geoboard.

Activities

Connections: Pythagoras on the Geoboard (continued)	Notes
Activity 2 (5 minutes) Ask participants to build a tilted square on the geoboard and determine the length of its side. **Activity 3 (10 minutes)** 1. Ask participants to identify two points on the geoboard that are not placed on the same row or on the same column. Ask them to join them with a rubber band, and to determine the distance between the two points. 2. Ask them to find a square having the marked segment as its side. Ask them to use the Pythagorean theorem to find the area of the square, and then find the square root of that number. **Inverse of Pythagorean Theorem** This is supplementary material for groups that move at a faster pace than other groups in the class.	This activity can be helpful to later understand how the distance between points in a coordinate system is computed.
Closure	
Have participants reflect on the session by asking them to share what was the highlight o today's activities.	
Take Home Activities	
Handouts of activities done in class are provided for participants to take home and do with their children.	
Preparation for the Next Session	
Collect name cards for use in the next sessions.	