A "Thought" Experiment

Approximating the circle by regular polygons.

Imagine you have a family of regular polygons inscribed in the same circle constructed in the following way. Starting with a regular hexagon (figure 1a), the next polygon will have 12 sides. Six of the vertices will be common with the hexagon; the additional vertices will be the midpoints of the arches (see figure 1b). In the same way, each successive term of the family of polygons has twice as many sides. The perimeters of the regular polygons approximate better and better the circumference of the circle. Furthermore, by using a polygon with enough number of sides, we can make the difference between the perimeter of the polygon and the circumference as small as we want. The areas of the regular polygons approximate better and better the area of the circle. The difference between the area of the circle and the area of one of the polygons can also be made as small as we want by choosing a polygon with enough number of sides.

The area of the regular polygon can be computed by multiplying the perimeter times the height of one of the triangles forming the regular polygon (see figure 2), and dividing by two. This can be proved in several ways. One is to imagine all the triangles laid-out side by side (figure 3). The total area of the polygon is the sum of the areas of the triangles. One method to obtain the total is to compute the area of each triangle by multiplying the base times the height, divide by 2, and then add the areas. Or we can add all the bases first, which gives us the perimeter, then multiply by the height, and divide by 2. If the number of sides of the polygon is very large, the sum of the bases will be very close to the circumference of the circle ($2\pi r$), and the height of the triangle will be very close to the radius (r). Therefore the area of the polygon will be very close to $\frac{\text{circumference x radius}}{2}$.

Because the area of the circle and that of the polygons can be made as close to each other as we want, we have the area of the circle given by:

$$\frac{\text{circumference x radius}}{2} = \frac{\text{diameter x } \pi \text{ x radius}}{2} = \frac{\text{2 x radius x } \pi \text{ x radius}}{2} = \pi \text{ x radius}^2 .$$