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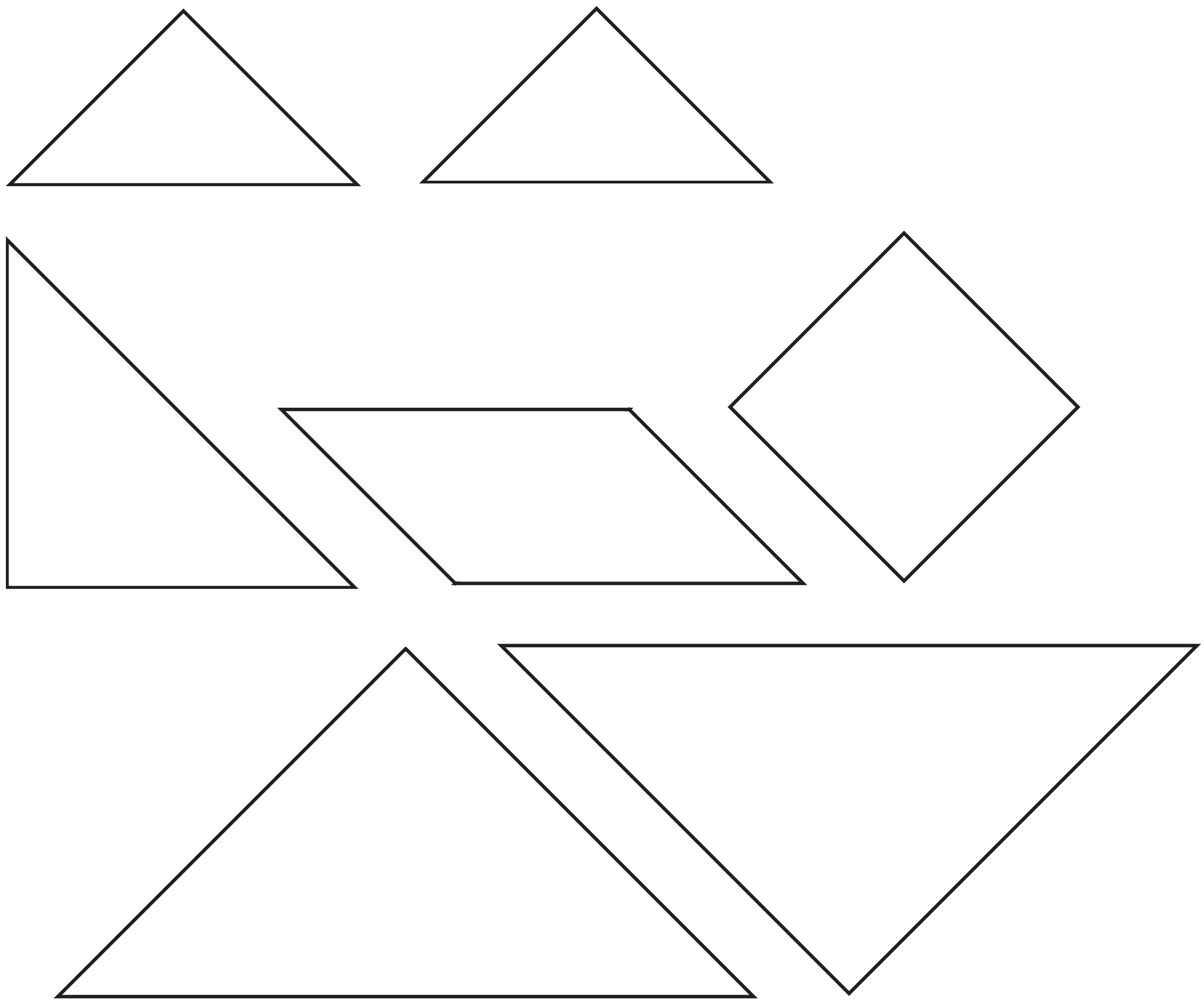
Tangram Puzzle Pieces

A tangram set is formed by seven pieces. These are provided for you to do the activity. However, you can copy this page, paste it onto cardboard and make your own set for home. Tangrams are also available commercially and they are quite inexpensive (an individual set can be ordered for about $1 from companies like Nasco).



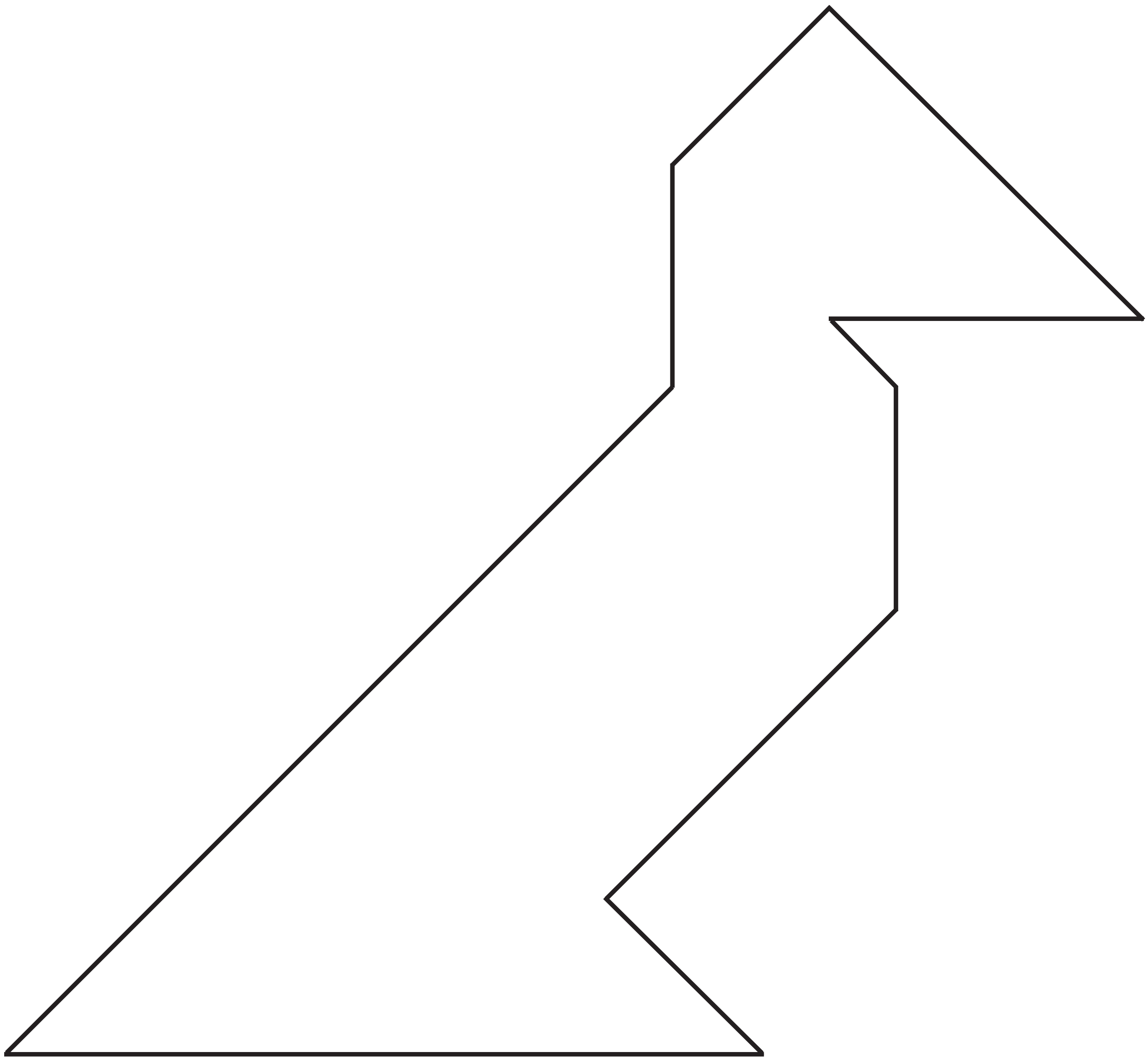
Children should have the opportunity to play first with the tangram pieces, forming puzzles such as the one shown below. By forming puzzles children will familiarize themselves with the geometrical shapes that form part of the tangram set. They will also learn how some of the pieces combined can form other shapes. They will also learn to see geometrical shapes embedded within other shapes. Tangrams can be used to develop an informal mathematical environment.

These activities for participants are the same kind that they can have with their own children. Participants should have the opportunity to try these activities first.

Shape of the Bird

Opening Activity

Use the seven tangram pieces to form the shape of a bird. Try this on your own before looking at the solution at the end of the handout.

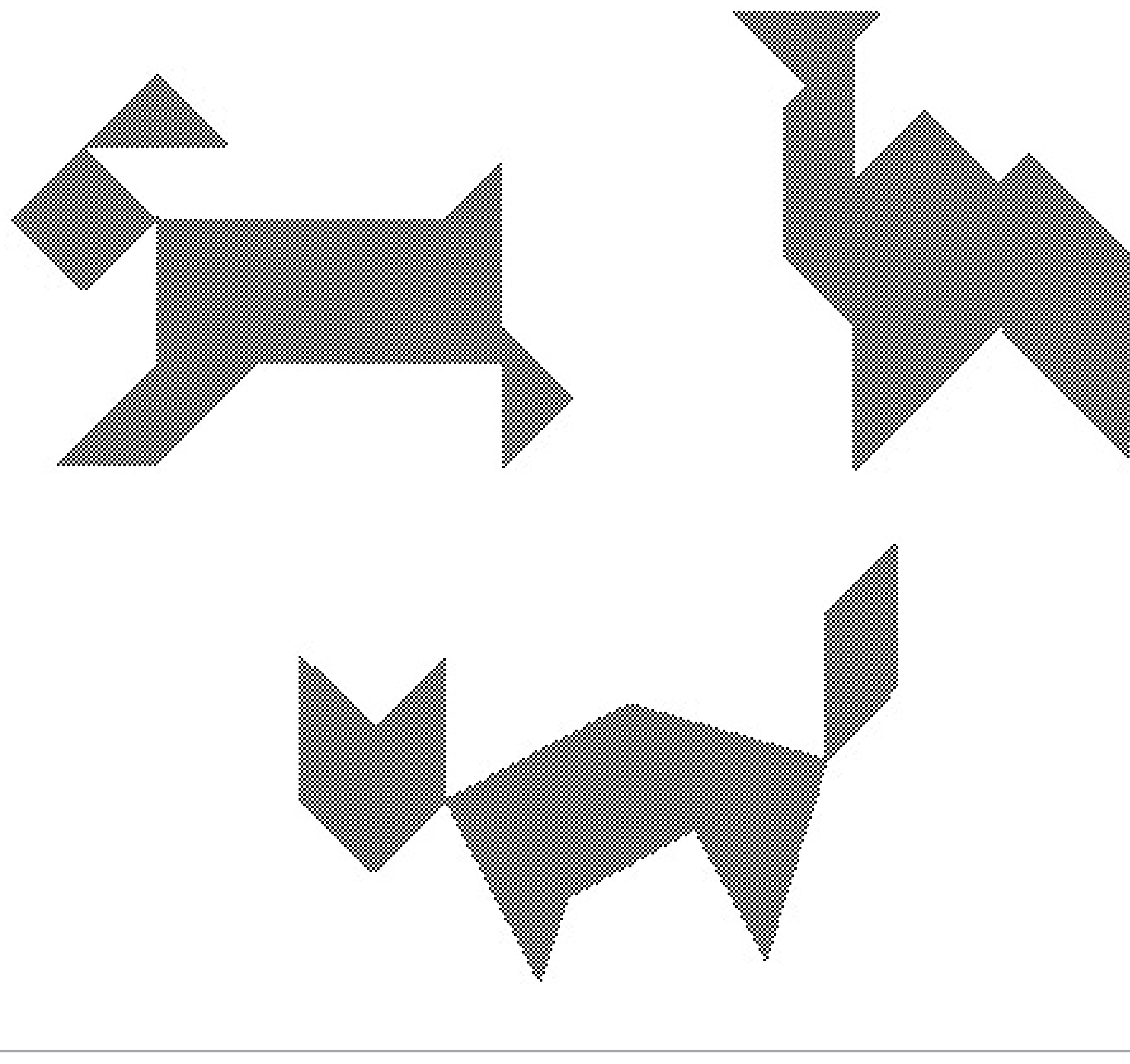


Further Activities with Tangrams

Tangram activities can be adapted for children of different ages. An activity for young children, (kindergarten and first grade) is to give them the outline of the shape to be filled with the tangram pieces, so that the solution is given. This activity helps children to recognize shapes, and match corresponding shapes in the solution. They will have to turn some of the pieces, and in some cases they will have to flip one of the pieces so that it fits.

Optional Activity

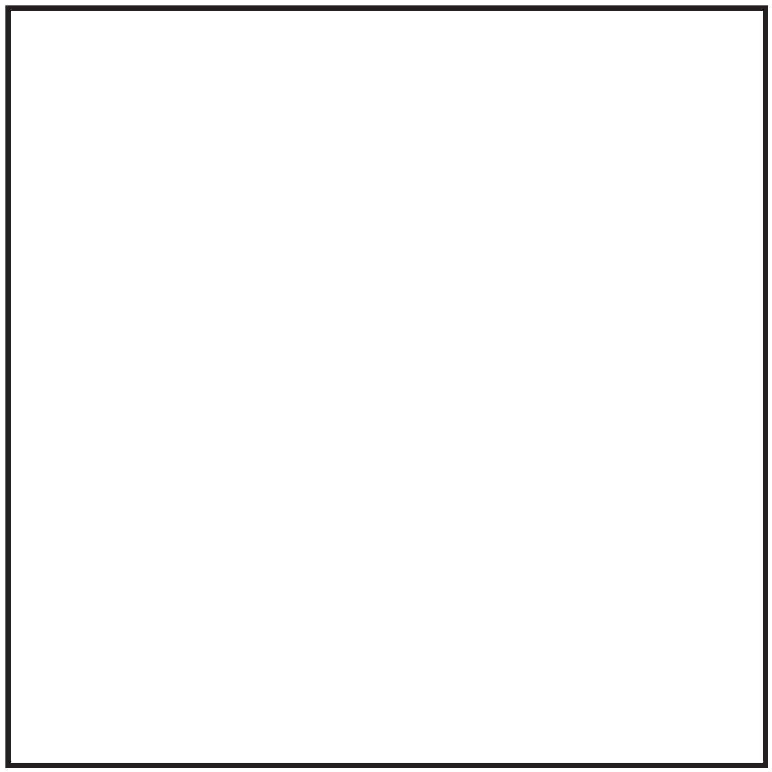
To make the activity more challenging, the outline of a shape can be given but at a different scale.



Geometry Explorations with Tangrams

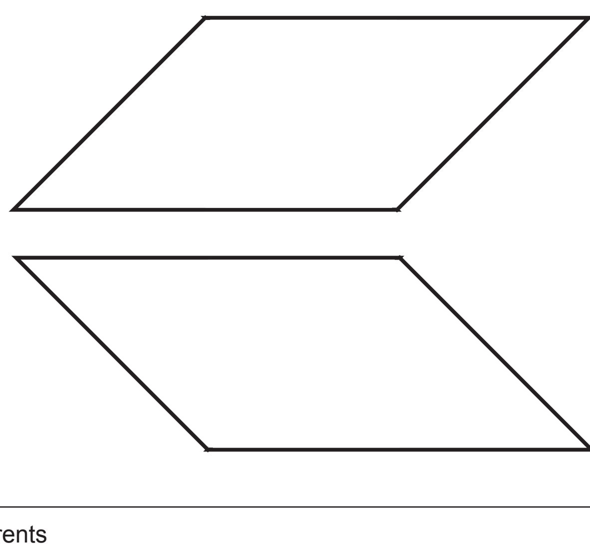
Activity 1

Fill in the outline for the big square tangram (all seven pieces are used). For older children this activity can be quite challenging by showing only the outline of the solution, so that the children have to figure out where the different pieces go inside. Try to fill the square yourself without looking at the solution at the end.



For small children in Kindergarten or first grade, an appropriate activity would be to provide the outlined solution as shown on the solution page and let them fill it with the pieces.

Notice that the parallelogram is the only piece in the tangram set that is not symmetrical. Therefore, sometimes you will have to flip their parallelogram to fit into a given parallelogram space.

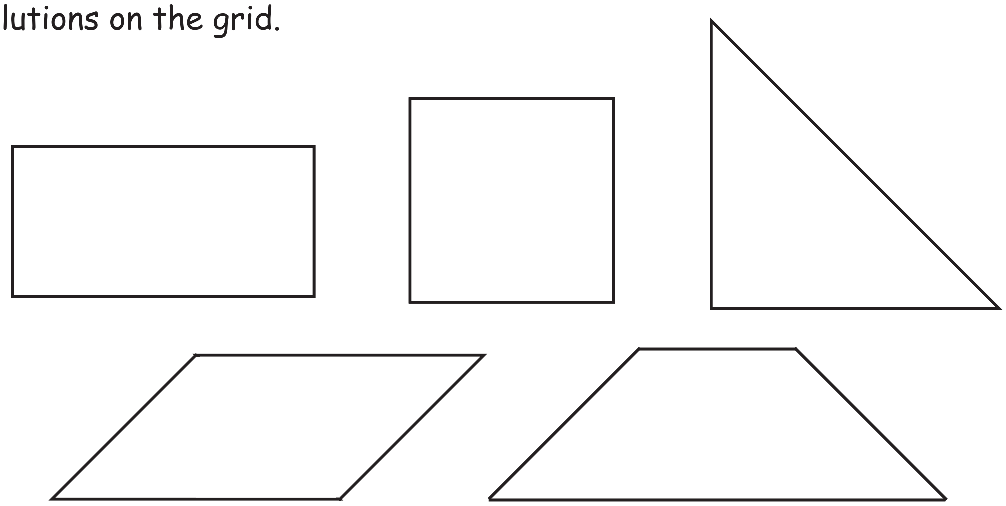


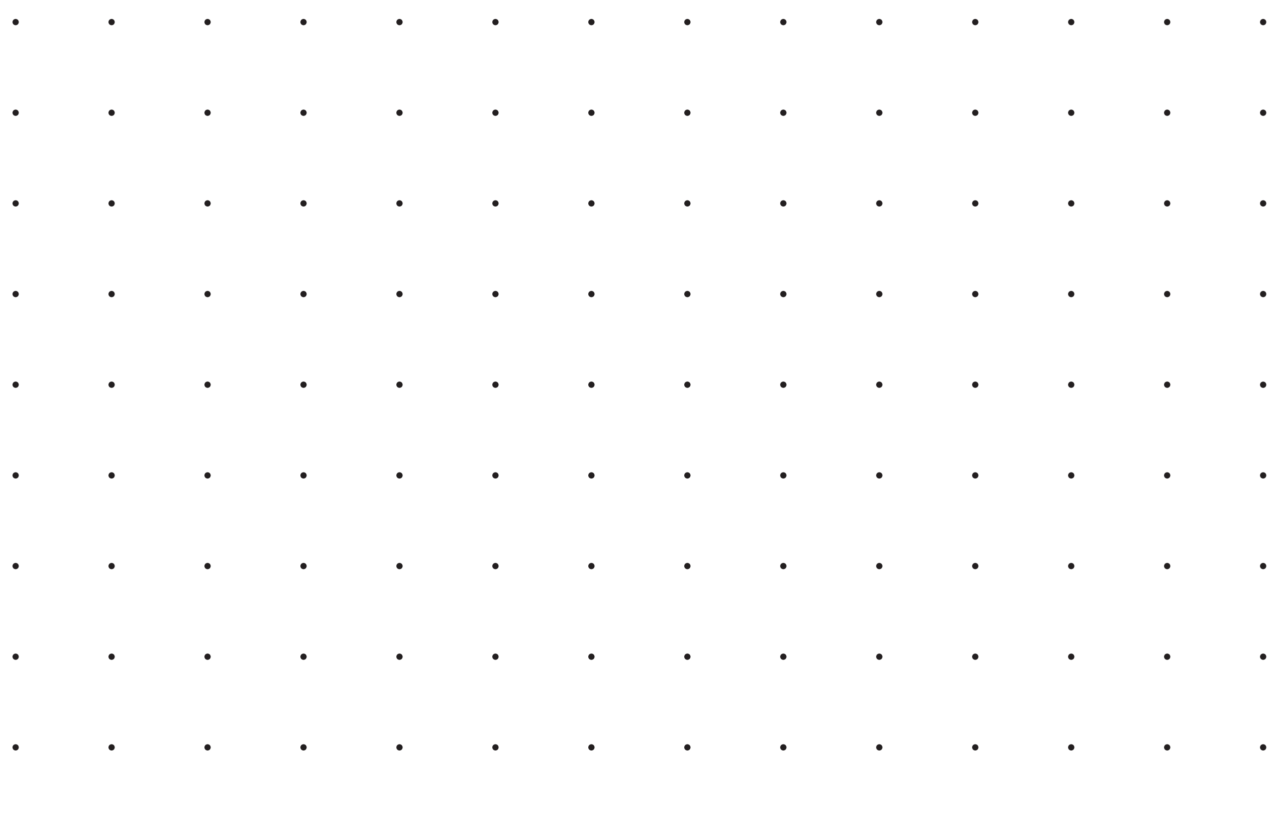
Geometry Explorations with Tangrams

Activities for upper elementary children (fourth and fifth graders) may include change of scaling. Children are given the shape to be built, but at a different scale. That is, the shape shown is not of the required size.

Activity 2

Using only the medium triangle and the two small triangles form the following shapes (the shapes are not shown at their actual size). Try on your own before looking at the solutions. Report your solutions on the grid.





Geometry Explorations with Tangrams

Activity 3

This is appropriate for students in 4 - 5 grade. Tangrams can also be used to develop problems where more than one solution is possible, or where a solution is possible only if we modify the condition of the problem.

Find whether you can form a square with exactly:

a) one piece
b) two pieces
c) three pieces
d) four pieces
e) five pieces
f) six pieces
g) seven pieces

For each case find as many solutions as possible. Use the grid to record your solutions.

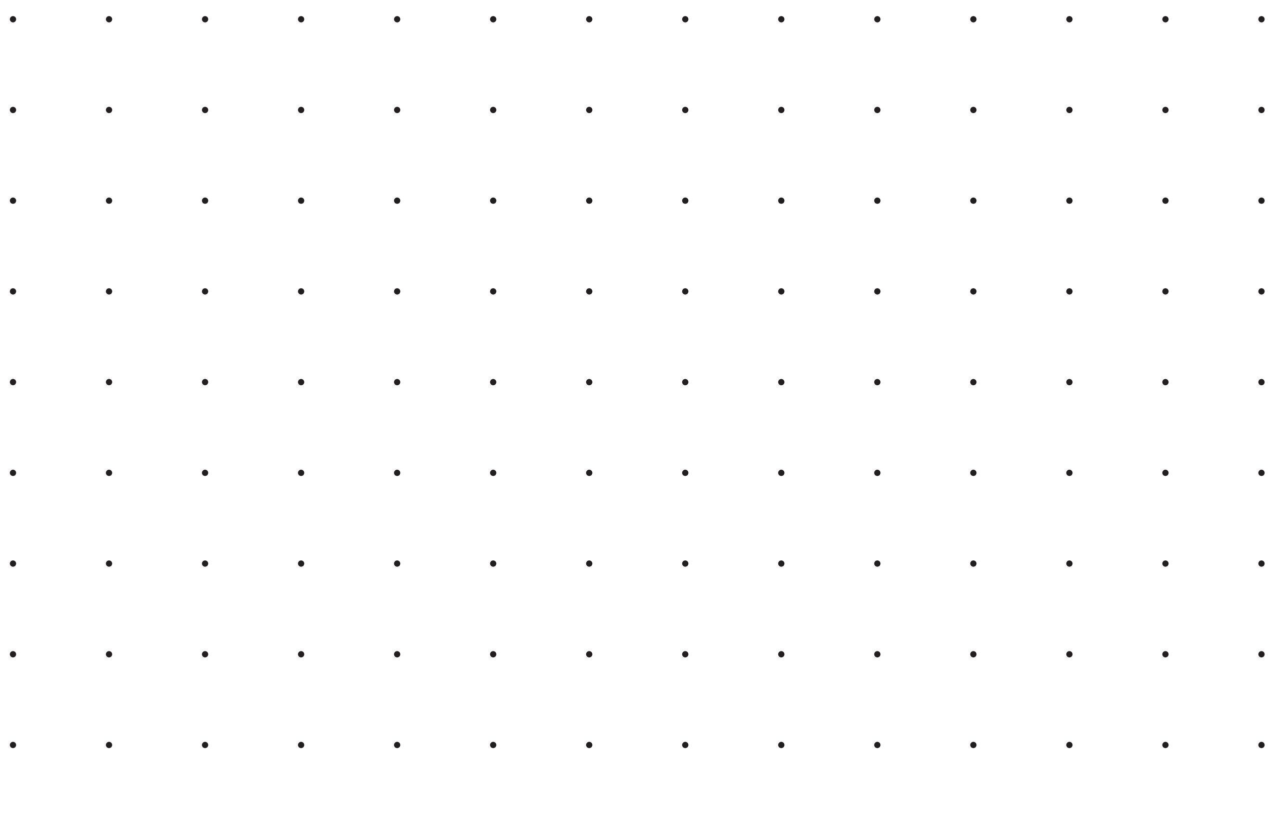
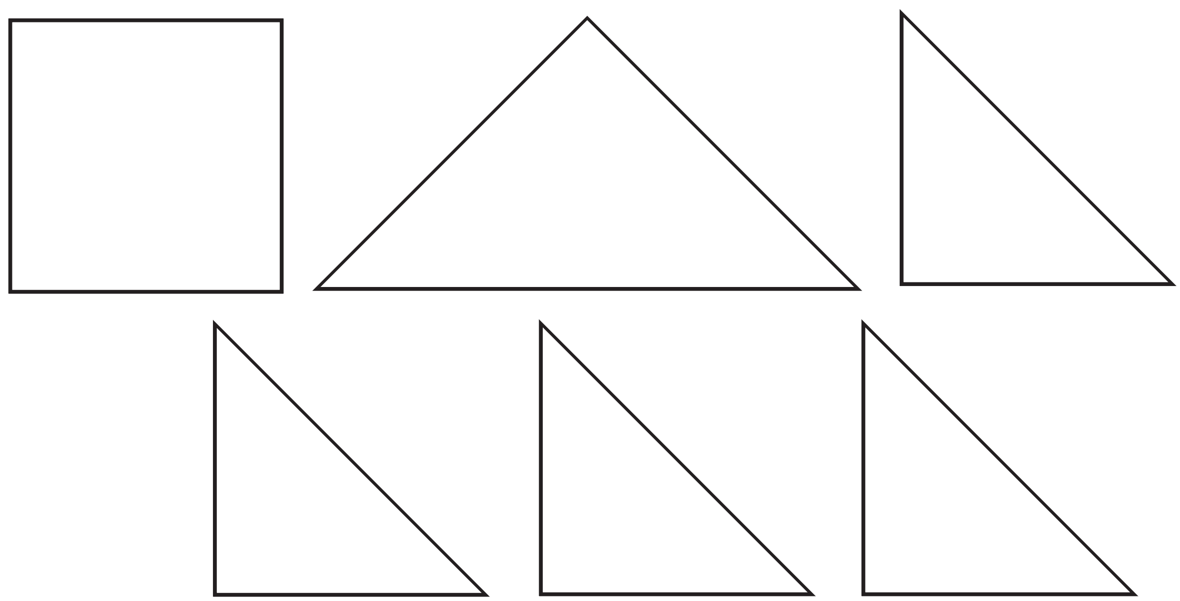


Figure 1
The set of cardboard figures used in these activities.



Hands-On Geometry

The activities described here are creative, informal, intuitive geometry activities to stress higher-order thinking. With these activities children develop spatial concepts such as conservation of area—if we cut a figure into two parts and rearrange the pieces we will obtain another figure that has the same area, even though it may look bigger. Children in kindergarten can do the first five activities in a session of 20-30 minutes. Informal language such as "the small triangle" can be used and accepted at this level. The number of activities can be increased for higher grades. Activities are also adjusted for each grade with more vocabulary added, and a more precise description of the figures. For the activities, right isosceles triangles of two sizes, and a square cut out from cardboard as shown are used.

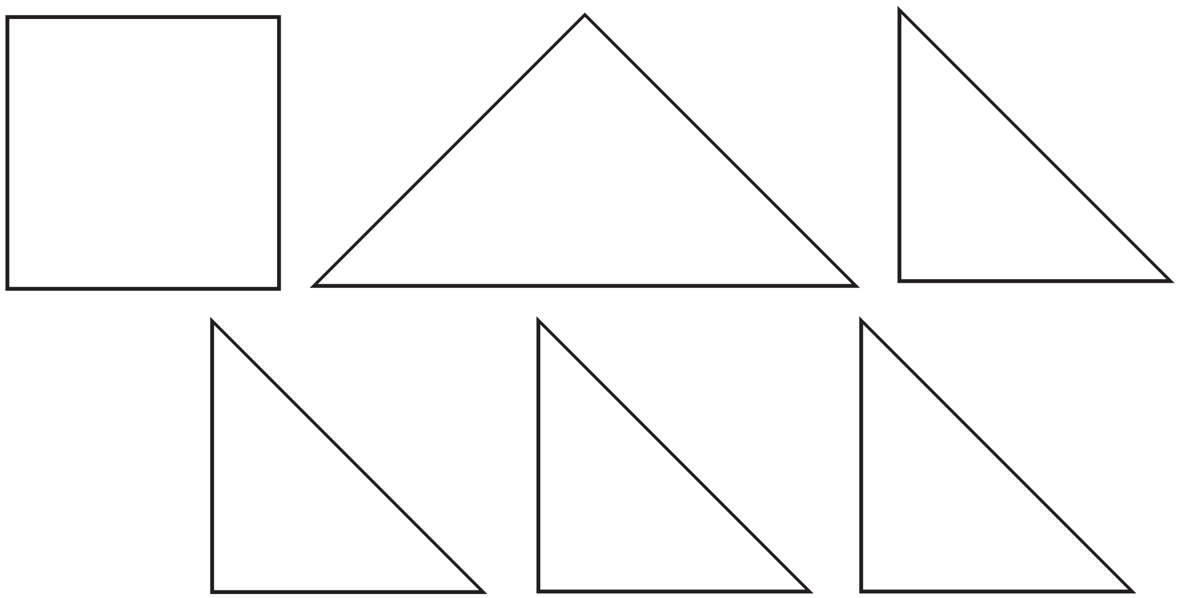


Figure 1. The set of cardboard figures used in these activities.

Activity 1

1) Take the cardboard square and cover the square in Figure 2.
   - *What can you say about the shape and size of the two shapes?*
   - *What can you say about their areas?*

2) Use the cardboard square and cover the shape Figure 3.
   - *What can you say about the two shapes?*
   - *What kind of shape is Figure 3?*

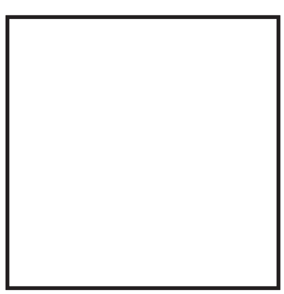


Figure 2

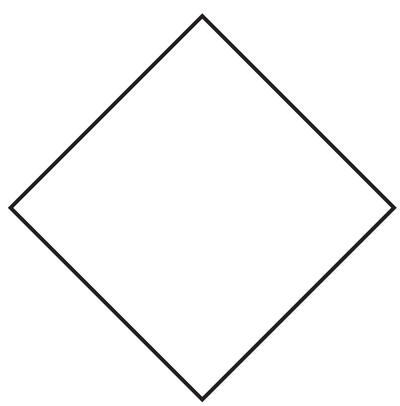


Figure 3

Hands-On Geometry

Activity 2

1) Take one of the small cardboard triangles, and cover the triangle in Figure 4.
   - *What can you say about the two triangles?*

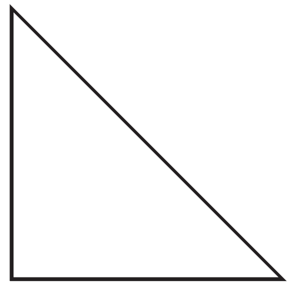


Figure 4

2) Next, cover each of the triangles in Figure 5 with the cardboard triangle.
   - *What can you say about all these triangles.*
   - *Does position and orientation alter the triangles?*

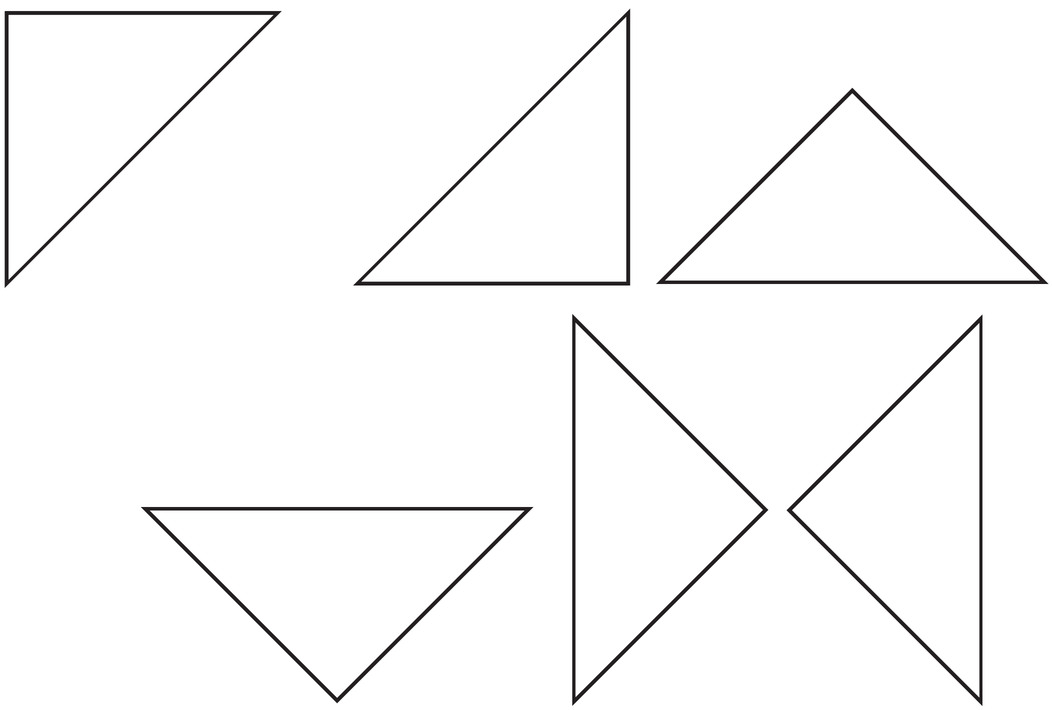


Figure 5

Activity 3

1) Use two of the small triangles to cover the square in Figure 2.
   - *What can you say about the area of the small triangle and the area of the square?*

2) Use the same two triangles to cover the square in Figure 3.

Hands-On Geometry

Activity 4

Take the big cardboard triangle, and cover each of the triangles in Figure 6 with it.

- *What can you say about all these triangles?*

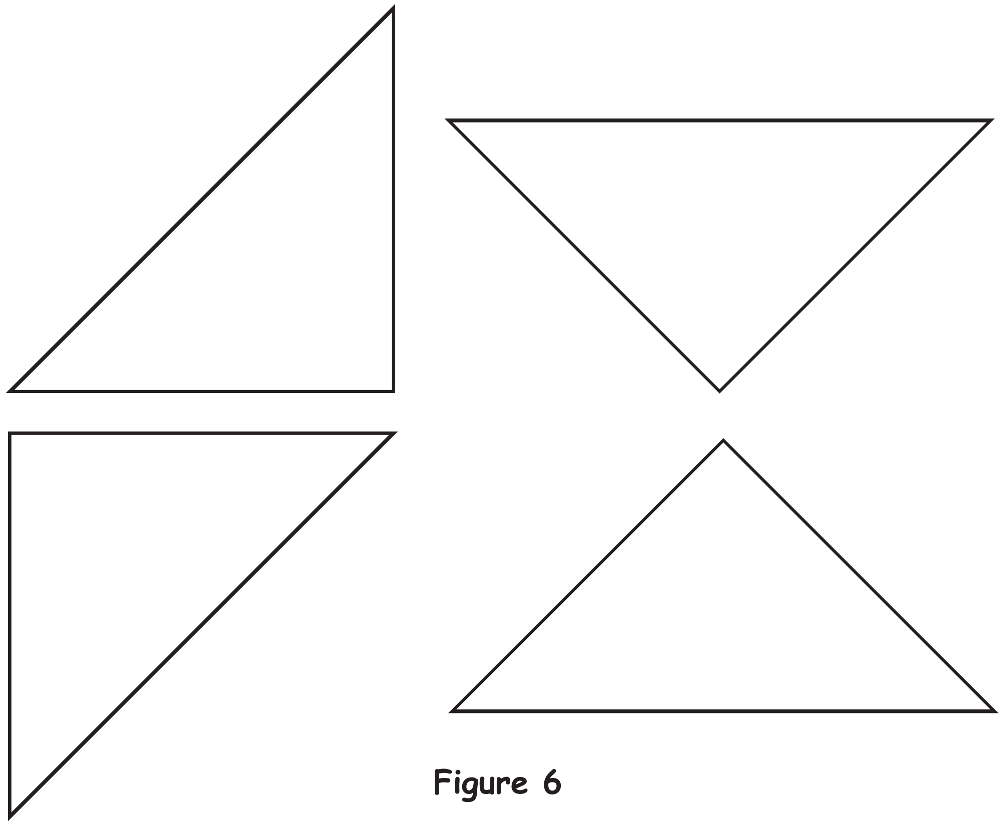


Figure 6

Activity 5

1) Use two small triangles to cover each of the triangles in Figure 6.
2) Compare the area of the small triangle with the area of the big triangle.

Activity 6

1) Cover the square (Figure 7) with two small triangles, and use the same two triangles to cover the big triangle (Figure 8).
2) Compare the area of the square with the area of the big triangle.

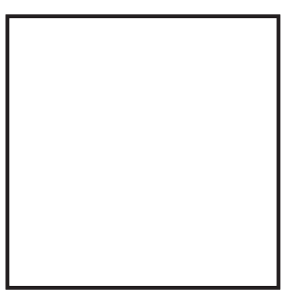


Figure 7

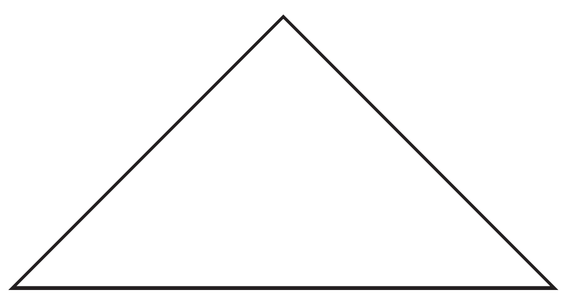


Figure 8

Hands-On Geometry

Activity 7

Cover the shapes in Figure 9 with two small triangles from Figure 1, to see that these shapes have the same area as the square.

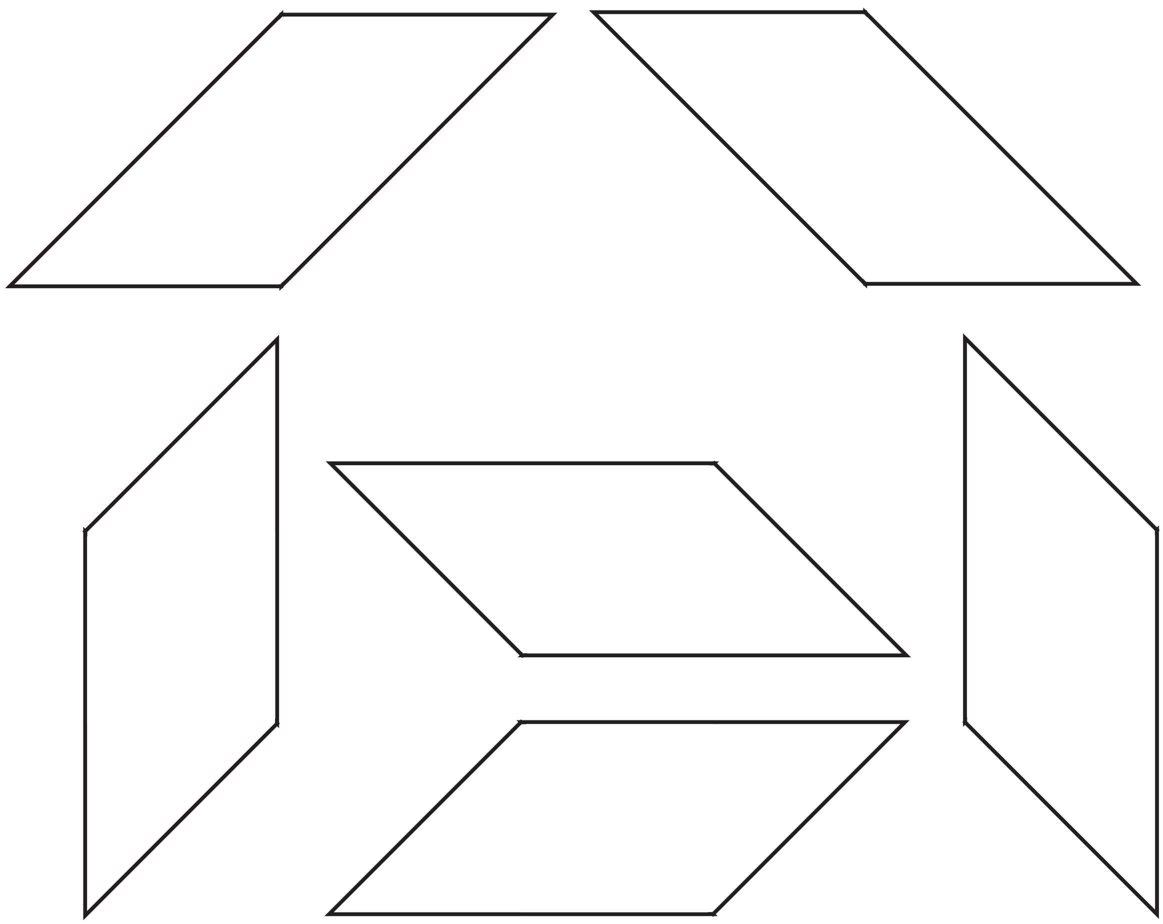


Figure 9

Hands-On Geometry

Activity 8

Cover the square in Figure 10 with four small triangles, and then the square in Figure 11.

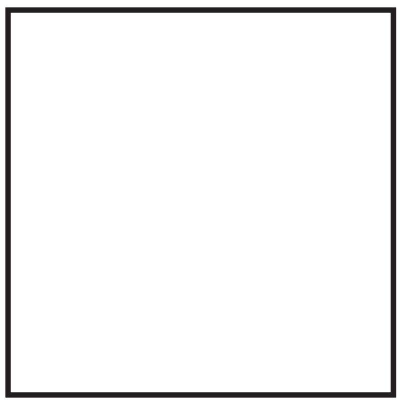


Figure 10

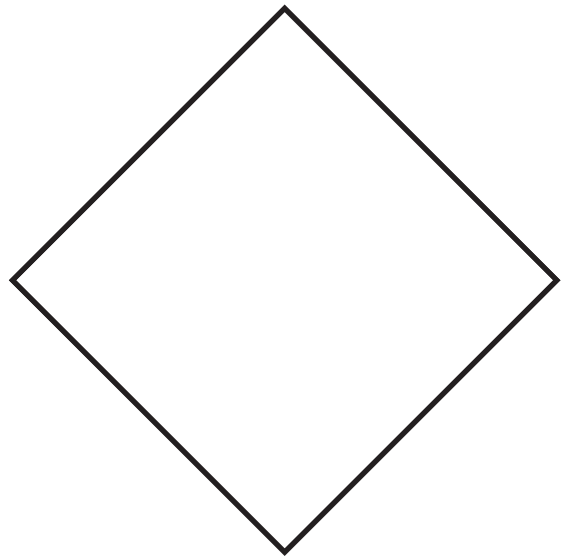


Figure 11

Activity 9

1) Cover each small square with two small triangles, and use the same four triangles to cover the big square.
2) Compare the area of the two small squares with the area of the big square in Figure 12.

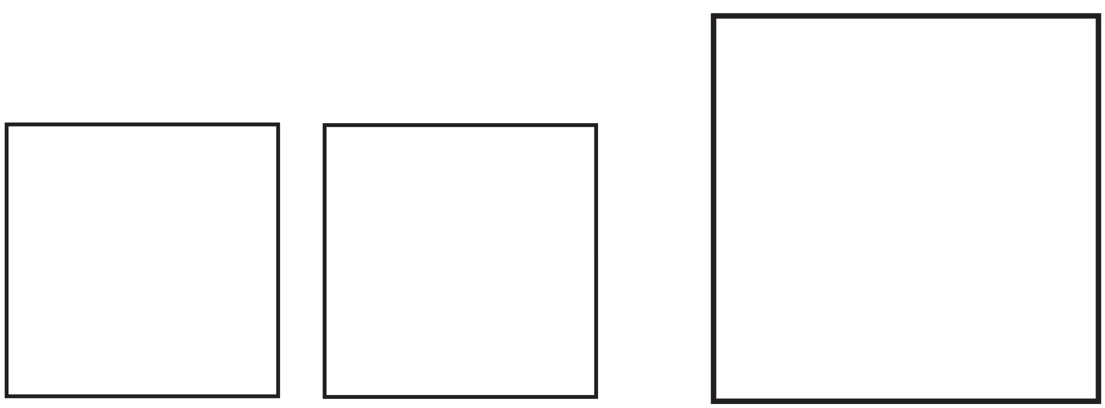


Figure 12

Hands-On Geometry

Activity 10

1) Use the four small triangles to cover the two small squares in Figure 13.
2) With the same four triangles they cover the big square.
3) Compare the areas of the two small squares with the area of the big square.

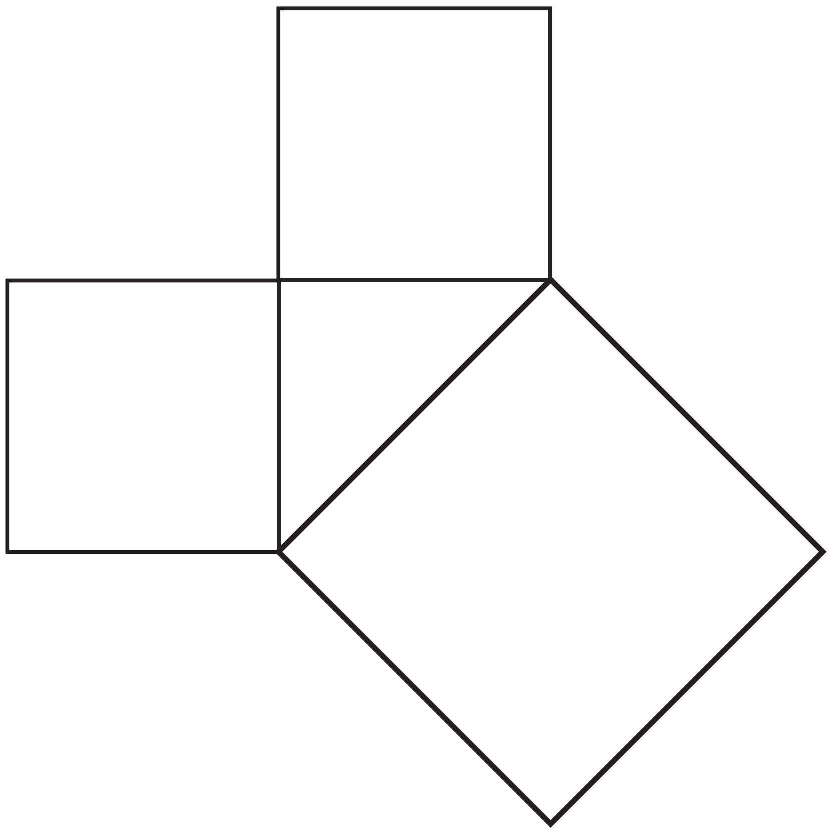


Figure 13

Verifying Properties of Squares

Patty paper that comes in squares can be used to focus the attention on parts of the square such as sides and angles and their properties. By folding the square and overlapping different parts we can see the relations between them.

Activity 1

Fold the square in half as shown in Figure 1, so that one side of the square overlaps the opposite side (a and d are opposite sides).

- *What can you say about the length of the opposite sides in a square?*
- *What can you say about the size of consecutive angles in a square?*

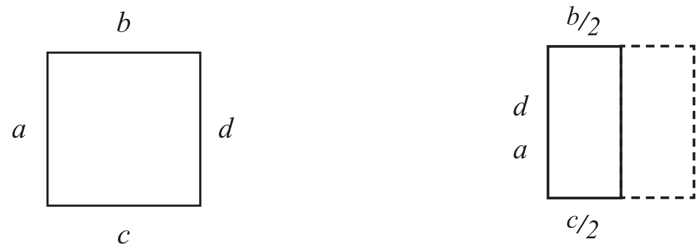


Figure 1

Activity 2

1) With a different square fold along the diagonal as sohown in Figure 2, so that one corner of the square overlaps perfectly on top of the opposite corner.
   - *What can you say about opposite angles?*
   - *What can you say about consecutive sides of a square?*

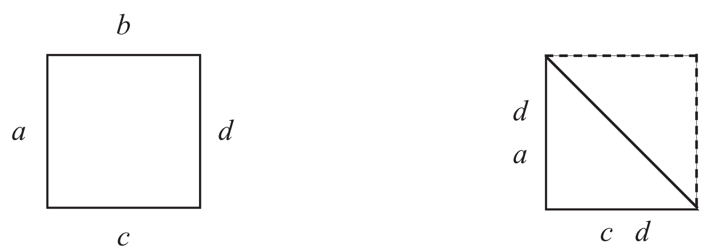
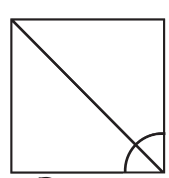


Figure 2

- *With the information we have so far can we say that all sides in a square are equal? Why?*
- *Can we say that all angles are equal? Why?*

2) Convince yourself that in a square the diagonal cuts the right angle in two equal angles.



Figure

Verifying Properties of Squares

Activity 3

1) Find the center of the square by folding the two mid parallels.
2) The two mid parallel lines form four smaller squares. Compare the area of each small square the area of the original square.

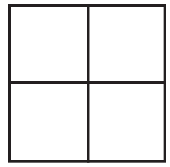


Figure 4

- *What can you tell about the angle formed by the mid parallels?*
- *What can you say about the lengths of the sides of the four small squares?*

Activity 4

1) Find the center of the square by folding the two diagonals.
2) Verify that in a square the diagonals cut each other in half.
3) Verify that the diagonals of a square intersect each other at a right angle.
4) The two diagonals form four right triangles.
   - *What can you say about the area of these four triangles?*

   Compare the area of one triangle to the area of the original square.

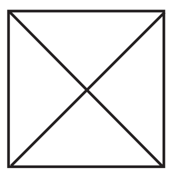


Figure 5

5) Verify that the diagonals and the mid parallels cross indeed in the same point.

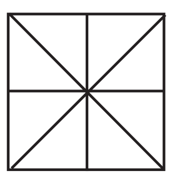


Figure 6

Activity 5

1) Compare the area of one of the triangles formed by the two diagonals with the area of one of the small squares formed by the to mid parallels.
2) Compare the shaded areas.

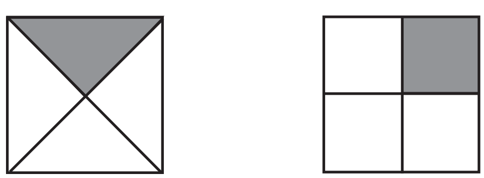


Figure 7

Verifying Properties of Squares

Activity 6

1) Find the center of the square by folding the mid parallels.
2) Now fold the corners of the square so that they meet at the center. Four right triangles and a square are formed.
3) Compare the area of this square to the area of the original square.

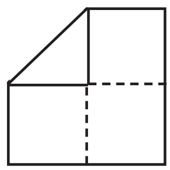


Figure 8
One corner folded to the center

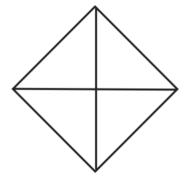


Figure 9
Four corners folded to the center

Activity 7

Mathematicians have agreed that the measure of a right angle is 90°.

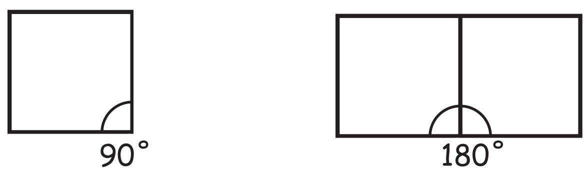


Figure 10

1) Verify that two right angles put together form a straight line. Therefore, the measure of a straight angle is 180°.

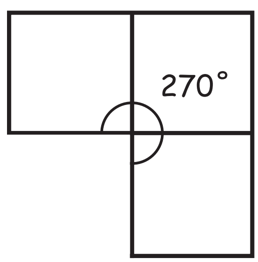


Figure 11
Three adjacent right angles.

2) Verify that four right angles put together fit exactly. The full turn is therefore 360°.

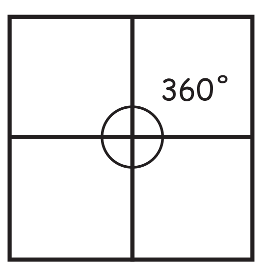


Figure 12
Four adjacent right angles

A Two-Smokestack Ship

Activities that involve folding paper to make ships and planes offer the opportunity for young children to develop intuitions and familiarity with geometric situations. Instructions and pictures to fold the ship can be found at http://www.public.asu.edu/~aaafp/barco/smkst.html

Start with a square sheet of paper. Find the center of the square by folding the middle lines and opening the sheet again. Notice that four squares are formed. Compare the area of each of these squares to the area of the original square.

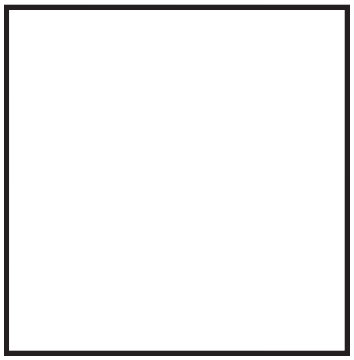


Figure 1. The original square

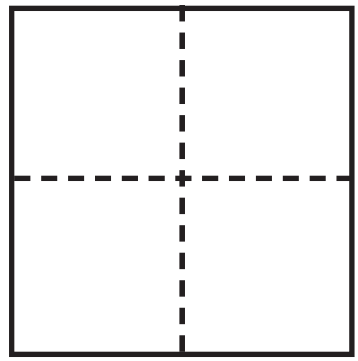


Figure 2. Two creases along the middle lines

Fold one corner so that the vertex is on the center (fig. 3). You obtain a right triangle. Fold the other three corners (fig. 4). You obtain one square formed by four right triangles. Compare the area of this square with the area of the original square.

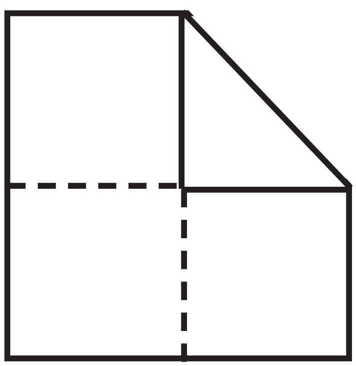


Figure 3

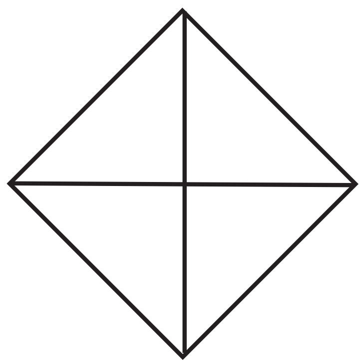


Figure 4

Flip the folded paper so that the other side is up. Fold one corner to the center (fig. 5).
Fold the other corners to the center (fig. 6). You again obtain a square. Compare the area of this square with the area of the original square.

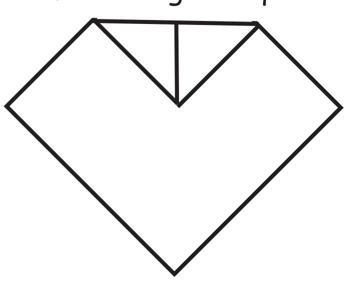


Figure 5

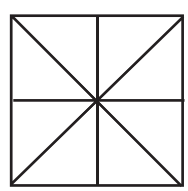


Figure 6

A Two-Smokestack Ship

Flip the paper again. You should have four flaps on the top side (fig. 7). Fold one corner to the center (fig. 8). Fold the other three corners (fig 9).

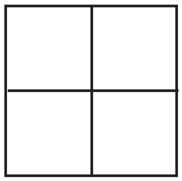


Figure 7

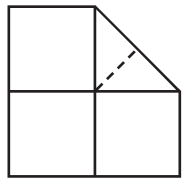
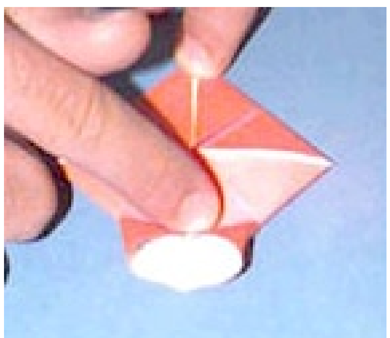


Figure 8


Figure 9

Flip the paper again. Open two of the squares, opposite to each other, to form the two smokestacks (fig. 10).


10a


10b

Figure 10. The smokestacks are formed

As you fold the ship to put the smokestacks side by side, push out the prow and the stern to finish the ship (fig. 11).

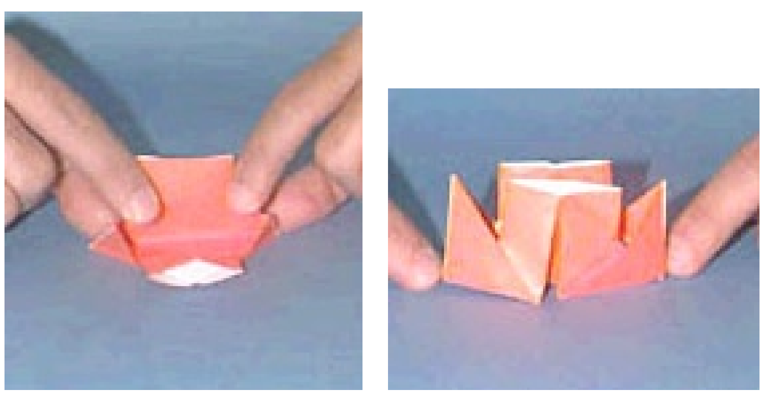


Figure 11

Van Hiele Levels

Level 1 Recognition. A student recognizes the shape as a whole. The student is not aware of specific parts or properties of that shape. The student identifies, names, compares and operates on geometric figures according to their appearance.

At this level, students may think that the two shapes in Figure 1 are different because they look different, not focusing on the sides or angles. Children may call the second shape a diamond rather than a square, because they look different.

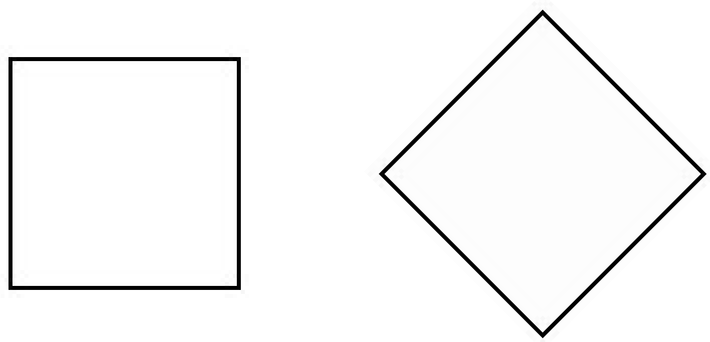


Figure 1

Level 2 Analysis of properties. A student can analyze figures in terms of their components, describe its parts and list its properties. Descriptions are used instead of definitions. Student discovers/proves properties or rules empirically (for example by folding, measuring, using a grid, or a diagram).

At this level, a student may realize that both shapes in figure 1 are squares, because they have four equal sides and four right angles.

Level 3 Informal deduction. A student can understand the role of definitions, the relationships between figures; can order figures hierarchically according to their characteristics; can deduce facts logically from previously accepted facts using informal arguments.

At this level, students will realize that a square is a special kind of rectangle, because it has four sides and four right angles, the defining characteristics of a rectangle.

Level 4 Axiomatic deduction. A student can understand the meaning of proof in the context of definitions, axioms and theorems. The student proves theorems deductively from axioms or theorems previously proven.

A few statements are accepted as axioms, that is, as self evident and without proof. All theorems are derived from these axioms or from previously proven theorems. This level corresponds to a traditional formal geometry course in high school.

Level 5 Rigor. A student can understand the relationships between different axiomatic systems. The student establishes theorems in different postulation systems and analyzes and compares these systems.

This level is attained only after students have had the opportunity to study both Euclidean and Non Euclidean geometries in an axiomatic treatment. This corresponds to advanced university mathematics.

Van Hiele Levels of Development in Geometry

Children's ways of thinking about geometrical objects change as they grow and have the appropriate learning opportunities. In this section we will describe the levels of development in geometrical thinking that people go through. It is important to realize that children's thinking in geometry does not automatically become more sophisticated as they grow older. In addition to maturation, instruction plays a major role. When students lack the opportunities to develop their thinking, they may remain at the same level. Many people become adults but they continue to use only basic ways of thinking about geometric objects. It is important that parents understand how children learn, and have examples of activities that can be used with children at various levels. However, the activities in this course will also help parents develop their own geometrical thinking.

According to Van Hiele (1986), there are five levels of development in geometrical thinking, which can be briefly described as follows (Fuys, Geddes, Tischler, 1988). In this course we will only present activities that correspond to the first three levels. Levels 4 and 5 are beyond the scope of this book and are described only for the sake of completeness. Level 4 corresponds to a traditional formal course in high school geometry.

See page 2 with Van Hiele Levels.

References

Fuys, D., Geddes, D., & Tischler, R. (1988). *The Van Hiele Model of thinking in geometry among adolescents.* National Council of Teachers of Mathematics.

Van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education.* Academic Press.

Note: Some authors (including Van Hiele) number the levels starting at 0. In this course we will use the numbering system as used in the Van Hiele Levels handout.

Van Hiele Levels of Development in Geometry

Level 1 Recognition. A student recognizes the shape as a whole. The student is not aware of specific parts or properties of that shape. The student identifies, names, compares and operates on geometric figures according to their appearance.

At this level, students may think that the two shapes in Figure 1 are different because they look different, not focusing on the sides or angles. Children may call the second shape a diamond rather than a square, because they look different.

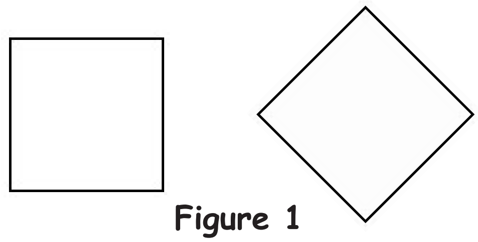


Figure 1

Level 2 Analysis of properties. A student can analyze figures in terms of their components, describe its parts and list its properties. Descriptions are used instead of definitions. Student discovers/proves properties or rules empirically (for example by folding, measuring, using a grid, or a diagram).

At this level, a student may realize that both shapes in figure 1 are squares, because they have four equal sides and four right angles.

Level 3 Informal deduction. A student can understand the role of definitions, the relationships between figures; can order figures hierarchically according to their characteristics; can deduce facts logically from previously accepted facts using informal arguments.

At this level, students will realize that a square is a special kind of rectangle, because it has four sides and four right angles, the defining characteristics of a rectangle.

Level 4 Axiomatic deduction. A student can understand the meaning of proof in the context of definitions, axioms and theorems. The student proves theorems deductively from axioms or theorems previously proven.

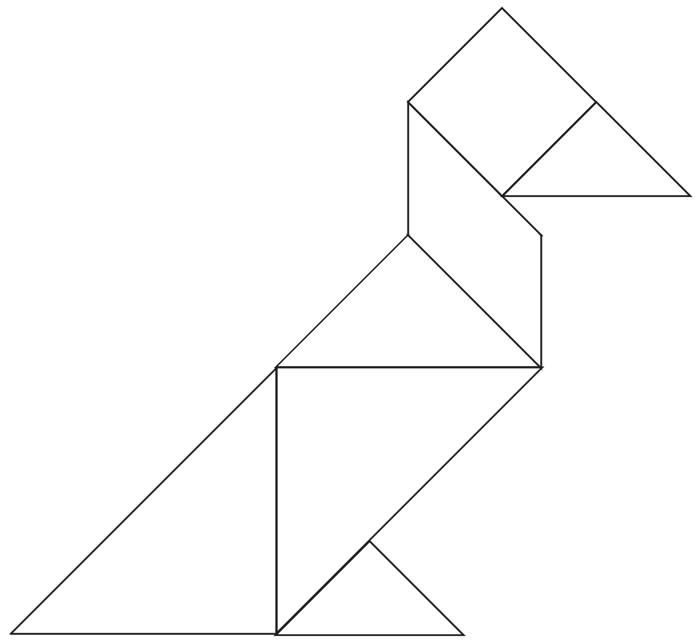
A few statements are accepted as axioms, that is, as self evident and without proof. All theorems are derived from these axioms or from previously proven theorems. This level corresponds to a traditional formal geometry course in high school.

Level 5 Rigor. A student can understand the relationships between different axiomatic systems. The student establishes theorems in different postulation systems and analyzes and compares these systems.

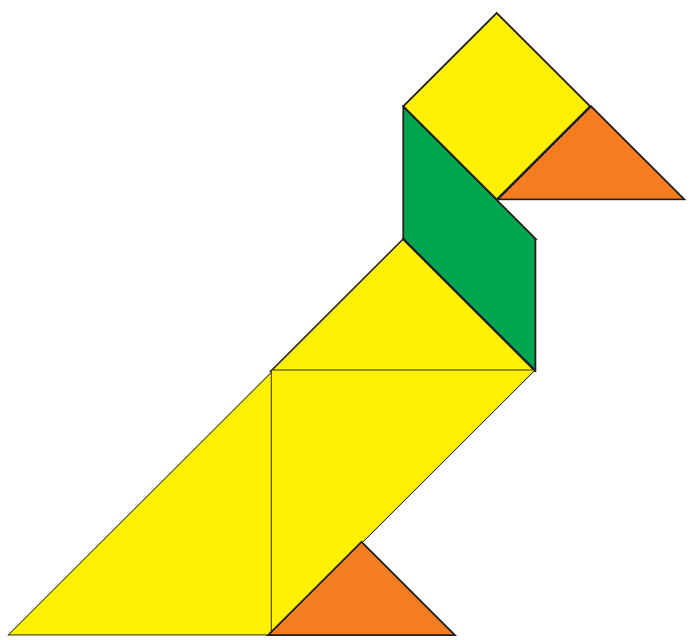
This level is attained only after students have had the opportunity to study both Euclidean and Non Euclidean geometries in an axiomatic treatment. This corresponds to advanced university mathematics.

Tangram Puzzle Solutions

Solutions to Opening Activity: Shape of the Bird

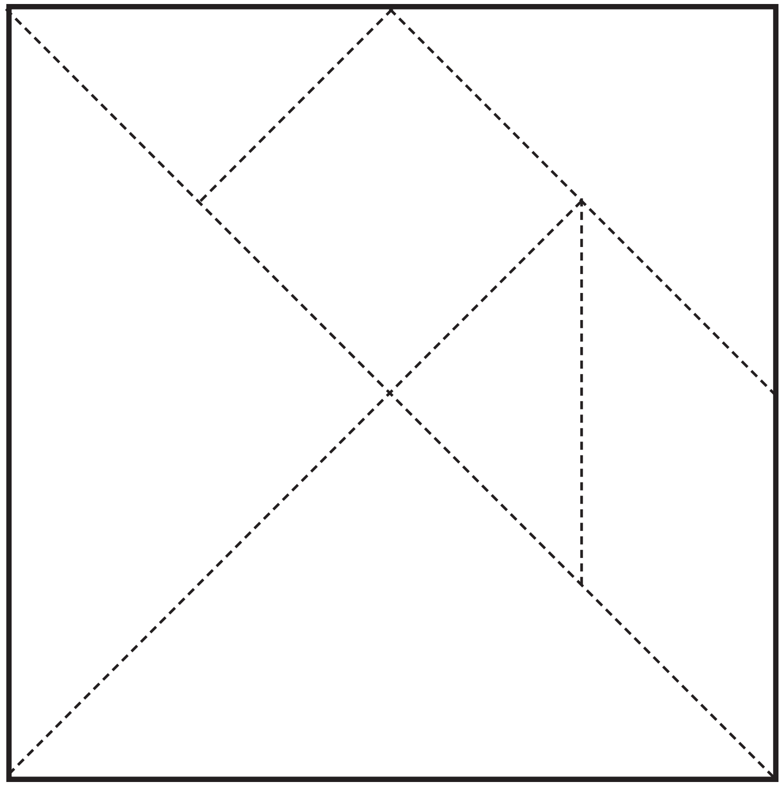


Children can use color to highlight the different pieces.
They can also use color to make the bird more beautiful.

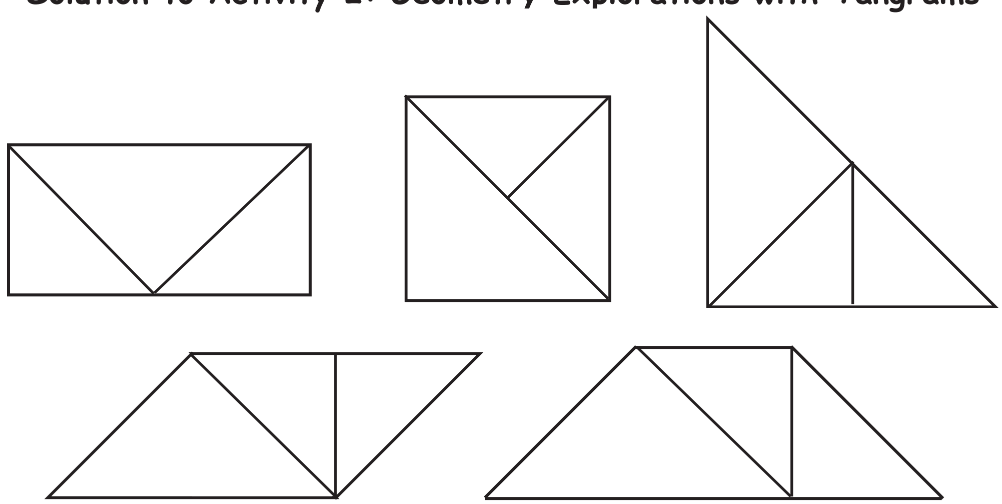


Tangram Puzzle Solutions

Solution to Activity 1: Geometry Explorations with Tangrams



Solution to Activity 2: Geometry Explorations with Tangrams

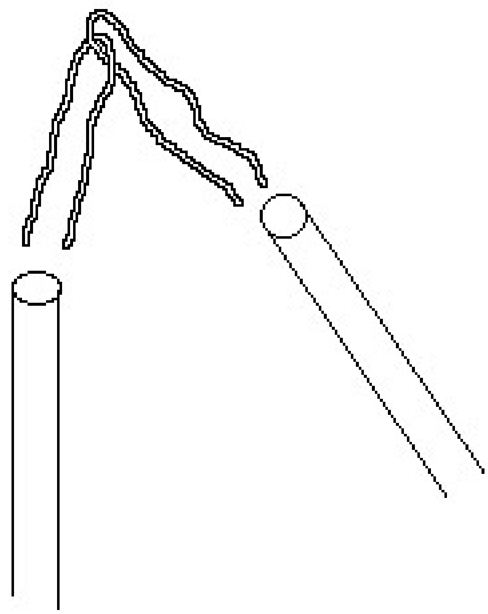


Note to Activity 3. All problems can be solved with one set of tangrams, except for forming the square with six pieces. To solve this problem, use six pieces from two sets of tangrams. Use the grid to report your solutions.

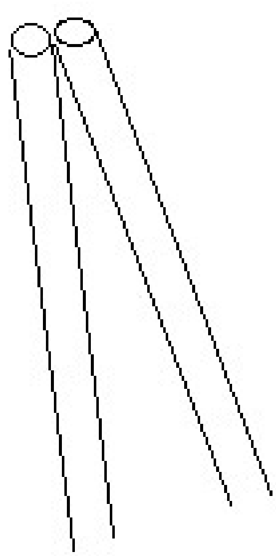
References
Cread, Ronald. *Tangrams—330 puzzles*. NY: Dover, 1980.
Stenmark, Jean Kerr, Thompson, Virginia, and Cossey, Ruth. *Family Math*. Berkeley, CA: Lawrence Hall of Sciences, 1986.

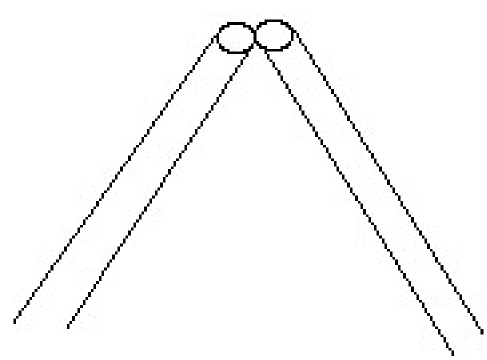
Angles with Straws and Bobby Pins

Join two straws with two bobby pins as indicated in the next figure. Notice that the bobby pins are intertwined.



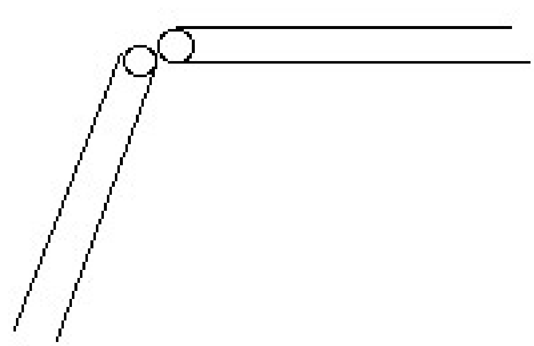
The straws will be the sides of the angle; the common point is the vertex of the angle. Start with the two straws close together. They form a very acute angle. The measure of this angle is small.





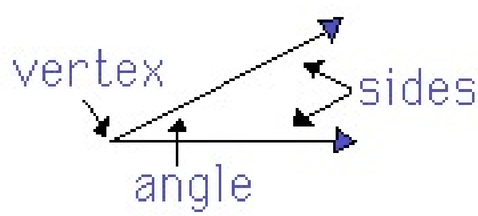
Angles with Straws and Bobby Pins

Open the angle to obtain a bigger angle. Keep opening until you form a right angle (the right angle is the angle of the corners of a rectangular piece of paper). If you keep opening the angle, you will obtain obtuse angles.

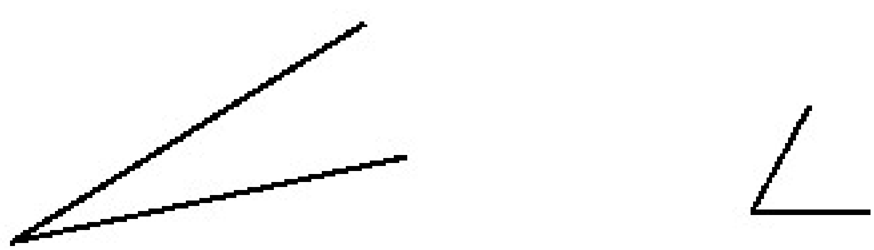


The parts of an angle

Two rays or segments that have a common vertex form an angle. The angle is the “opening” between the sides.



The size of the angle depends on how wide the opening is, not on the lengths of the sides. The angle to the right is bigger than the angle to the left.

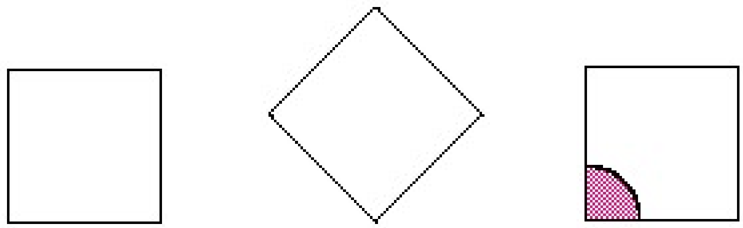


small angle bigger angle

Angles with Pattern Blocks

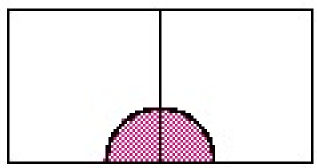
Activity 1. The Square

A square has four equal sides and four equal angles. The angles of a square are 90°.

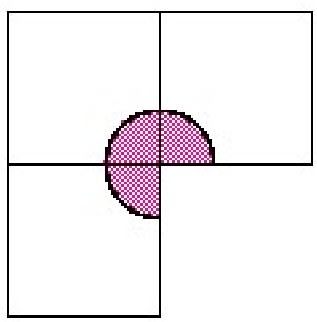


Squares A 90° angle

We will now look at how angles of multiple squares fit together around a common point.

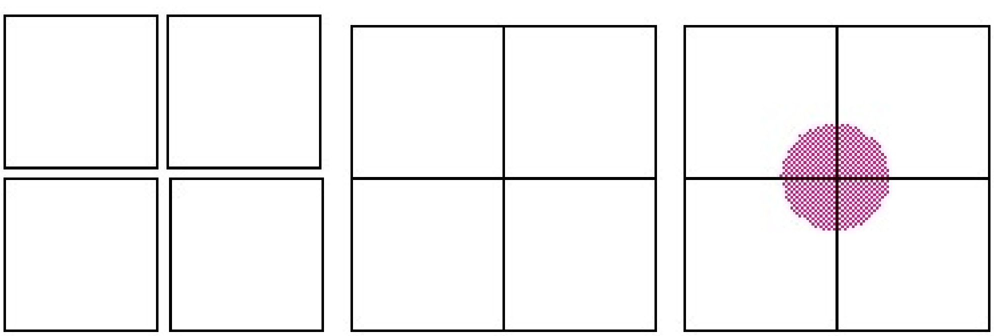


Two right angles together form a straight line. They add to 180°.



Three angles together add to 270°

If we put four squares around a point they fit perfectly, and so we can see that a whole turn around a point is 360°.

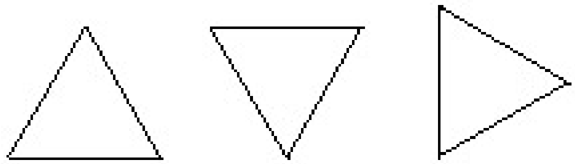


A 360° angle

Angles with Pattern Blocks

Activity 2. The Equilateral Triangle

An equilateral triangle has three equal sides and three equal angles.

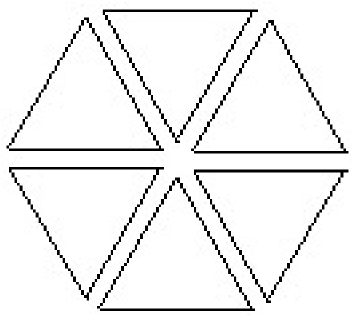


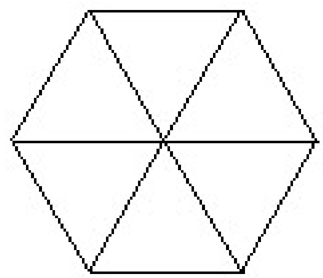
Equilateral triangles

In the same way, we can fit six equilateral triangles around a point.

What is the size of the angles in an equilateral triangle?

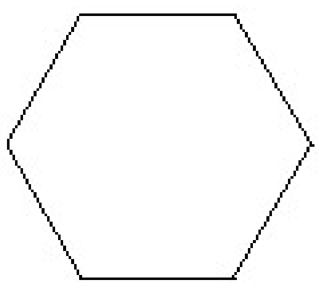
Justify your answer.



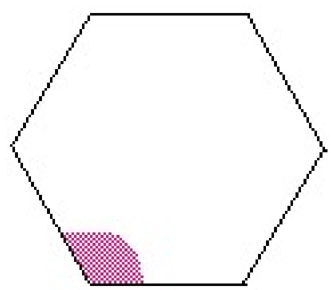


Activity 3. The Regular Hexagon

A regular hexagon has six equal sides and six equal angles.



Regular hexagon



One angle of the hexagon

How many regular hexagons can you fit around a point?

What is the measure of the angles of the regular hexagon?

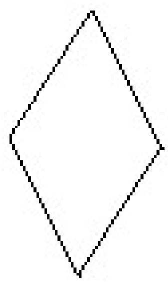
Justify your answer.

Relation with the Equilateral Triangle

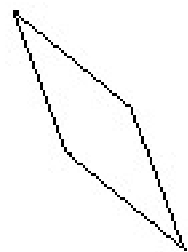
Describe how you can use two equilateral triangles to determine in a different way the size of the angles of a regular hexagon.

Activity 4. Two Rhombuses

Determine the angles in the blue rhombus and in the light tan rhombus. Provide convincing arguments that go beyond perception (it looks like...). You may overlap angles to make comparisons, or you can find other ways to convince yourself and others.



Blue rhombus

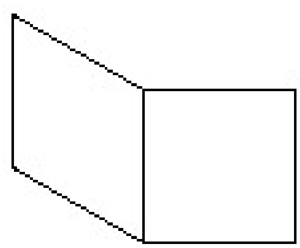


Light tan rhombus

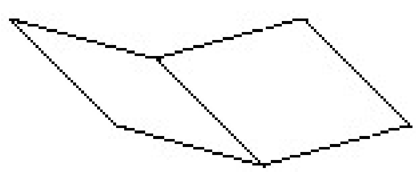
Activity 5. Placing other pieces together

Describe what angle is formed by placing together

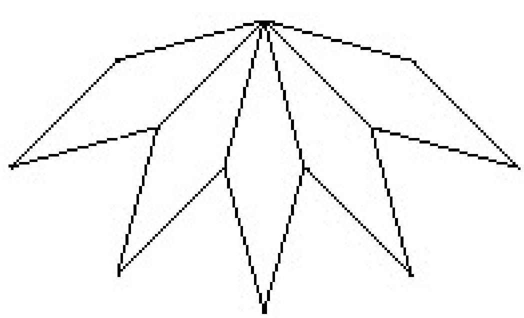
a) a square and a blue rhombus



b) a blue rhombus and a tan rhombus

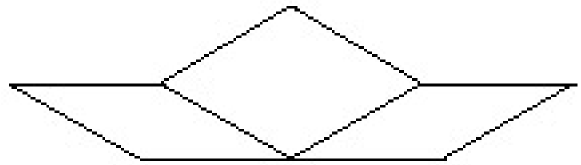


c) five tan rhombuses



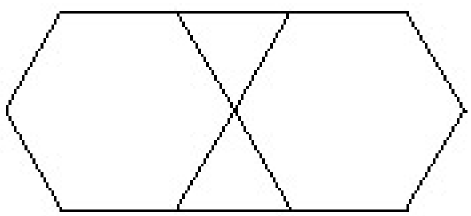
Angles with Pattern Blocks

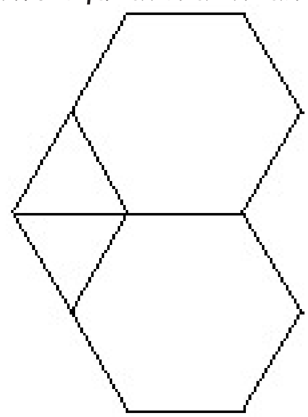
Show that you obtain a 180° angle by placing two light rhombuses on the sides of a blue rhombus.



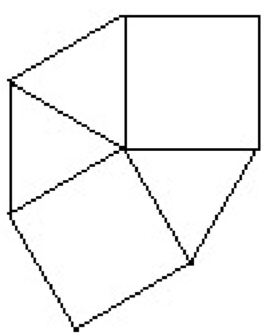
Activity 6. Verifying sums of angles

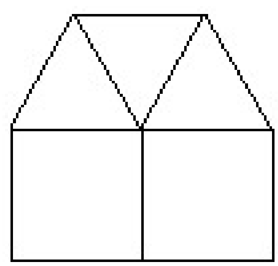
Give a convincing argument (beyond perception), that you can form a 360° angle with two regular hexagons and two equilateral triangles.





Give a convincing argument that you can form a 360° angle with three equilateral triangles and two squares.

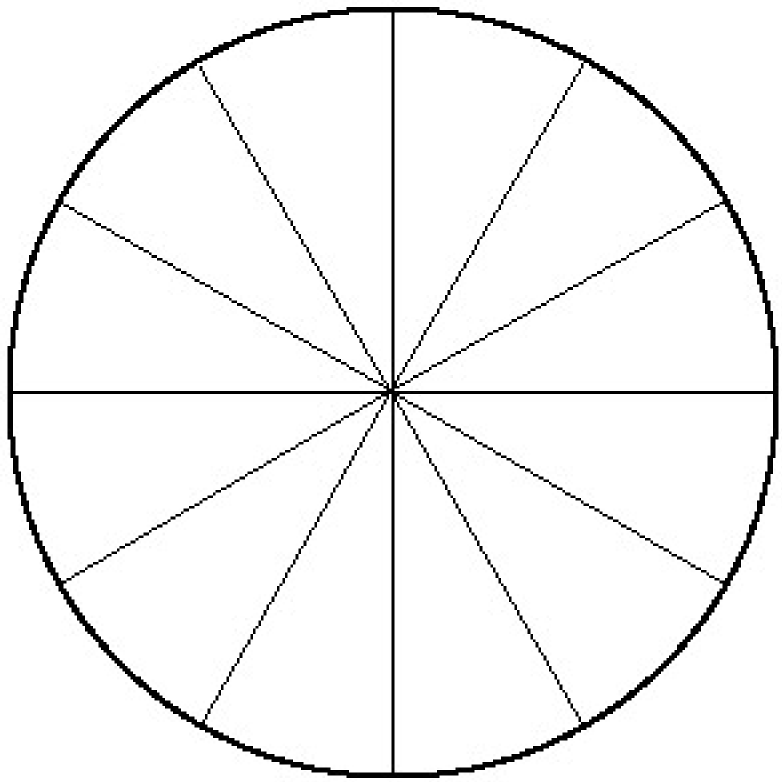




Measuring Angles with 30º Wedges

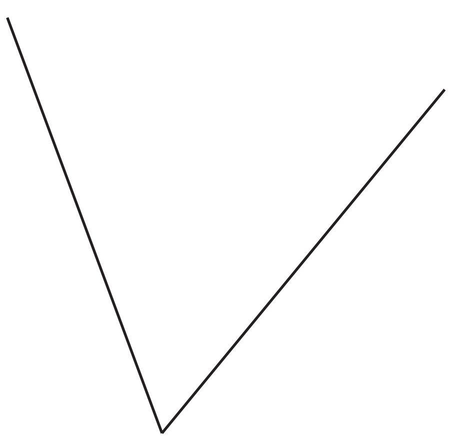
30˚ Wedges

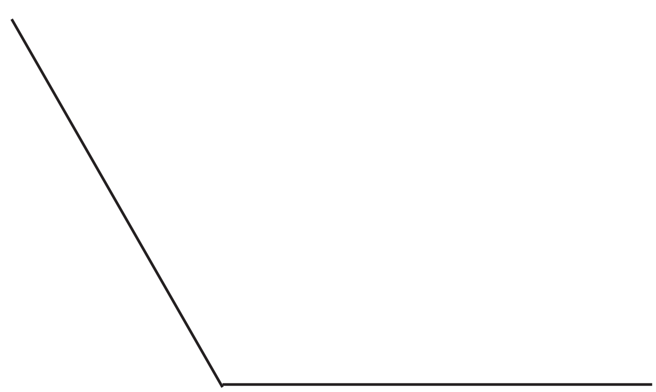
Angles can be measured by using 30˚ wedges. You can obtain your own set of wedges by pasting the circle on cardboard or copy on cardstock and cut the wedges out.



30˚ wedges

Determine the size of the following angles using the 30˚ wedges. Place the wedges inside the angles so that wedges and angles share vertices and wedges are placed next to each other without overlapping, and without leaving empty space between them.





Measuring Angles with 30º Wedges

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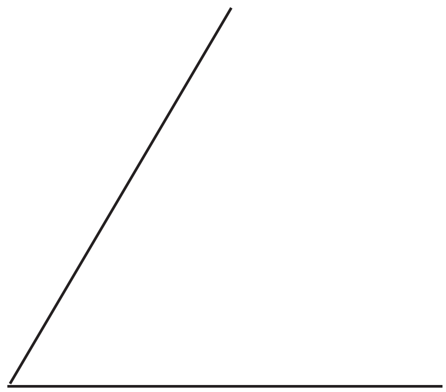

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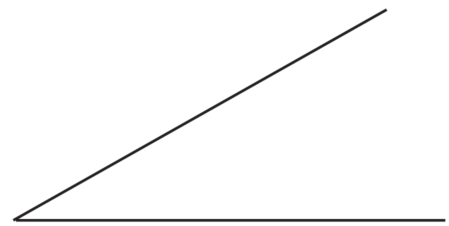

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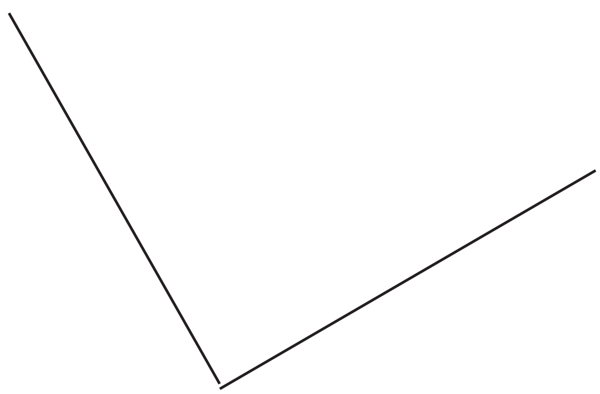

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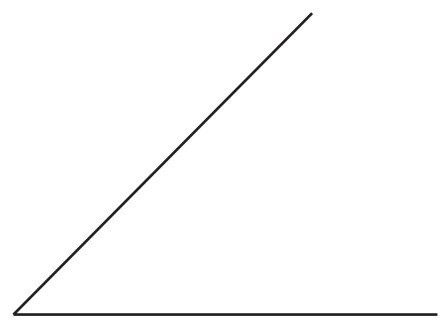

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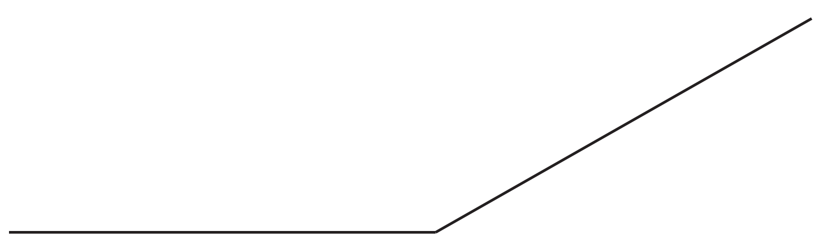

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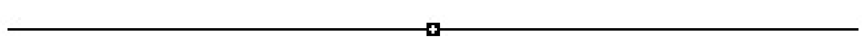
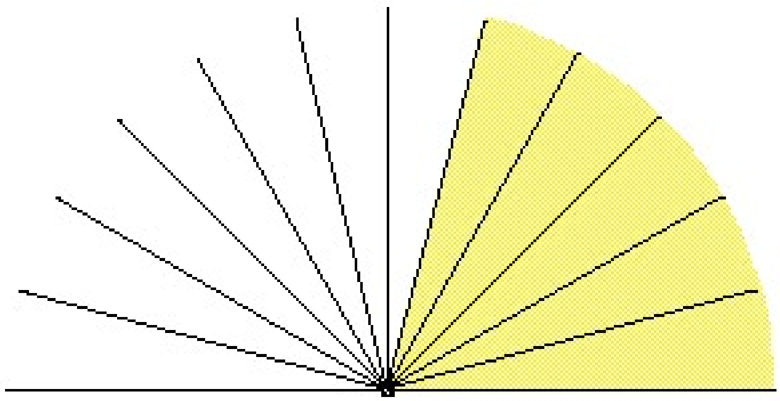
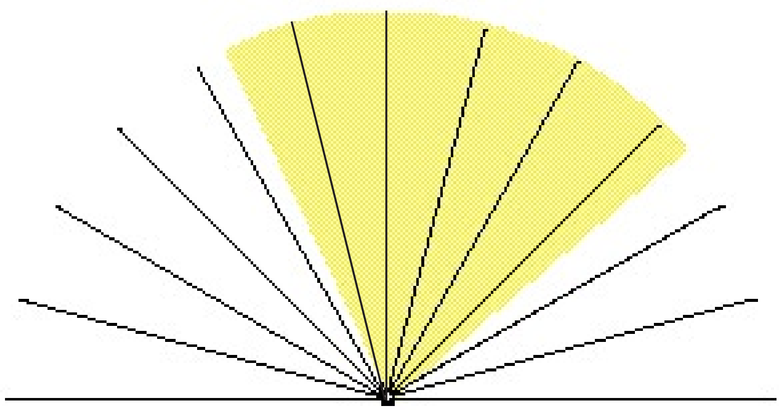


Using 15° protractor

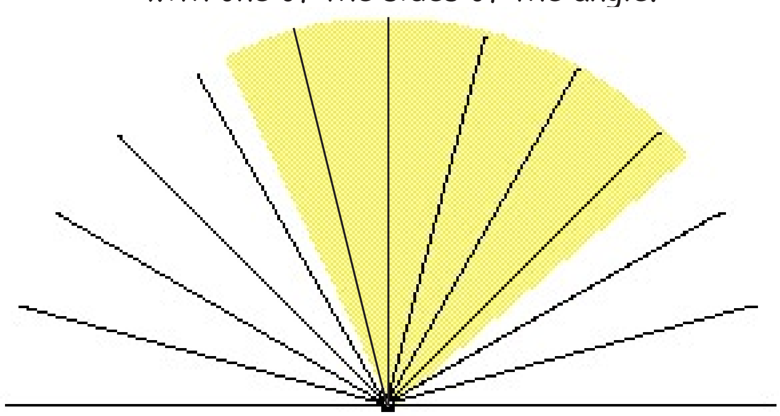
To measure angles it is important that the protractor is placed so that the central point is at the vertex of the angle that will be measured, and that one of the end lines of the protractor coincides with one of the sides of the angle.



Correct position to measure a 75° angle.



Incorrect: neither end line of the protractor coincides with one of the sides of the angle.

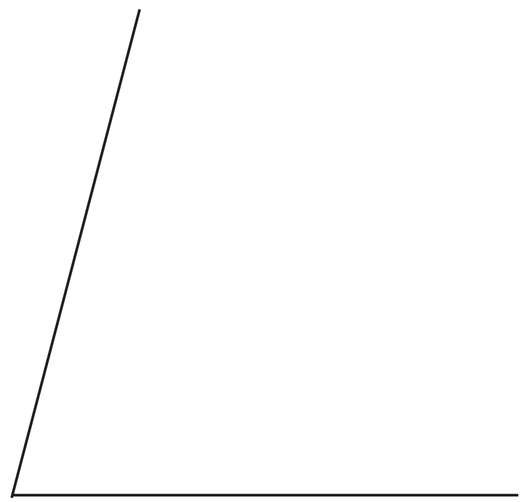


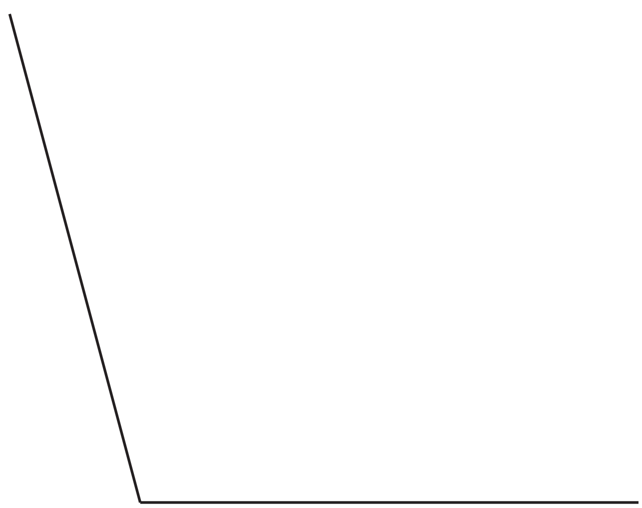
Incorrect: The center of the protractor is not at the vertex of the angle.

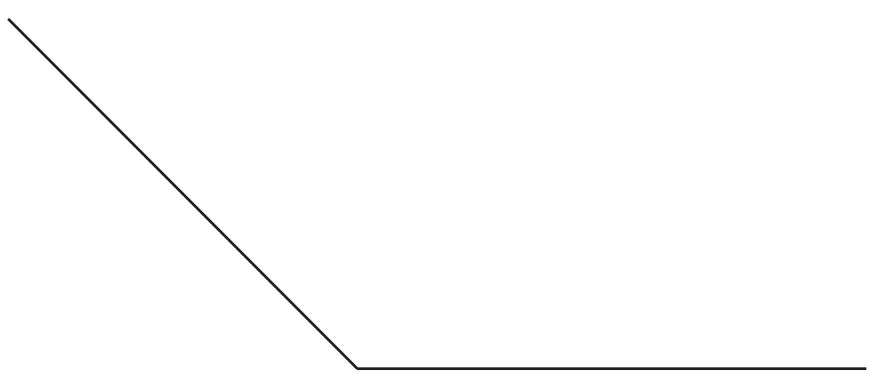
Measuring Angles with 15º Protractor

Using 15˚ protractor

Use 15˚ Protractor to measure the angles given in the previous activity and to measure the angles given below.



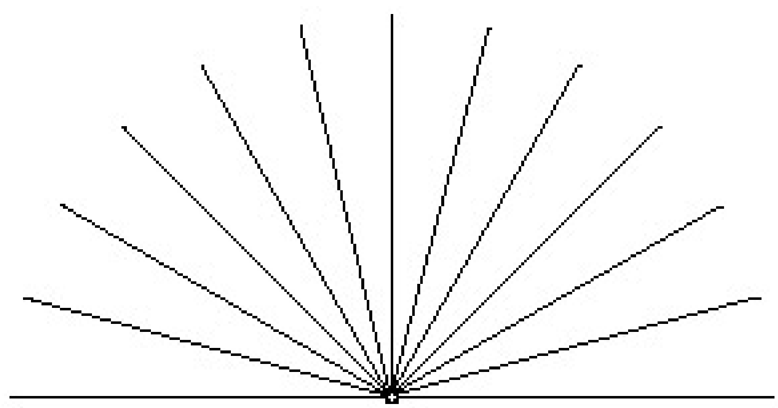




Measuring Angles with 15º Protractor

A simple protractor

You can make your own 15° protractor by copying the protractor onto a transparency.



Benchmarks for Angles

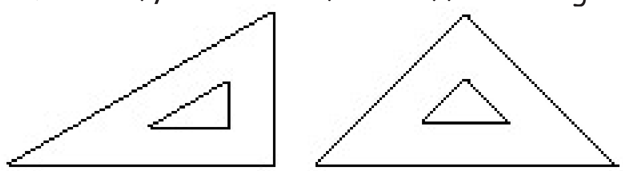
In the same way that we develop a feeling for distances and lengths using particular values as reference, we can develop a feeling for angles using particular values as references. Useful angles to have as a reference are 90°, 60°, 45°, and 30°. Combining these angles, we can also find benchmarks for angles bigger than 90°. Useful benchmarks are 120°, 135°, 150°, and of course 180°.

Activity 1

Describe the following benchmark angles using your own words: 0°, 30°, 45°, 60°, 90°, 120°, 135°, 150°, and 180°. For example, 45° is half a right angle; 60° is the angle of an equilateral triangle; 120° is the angle of a regular hexagon.

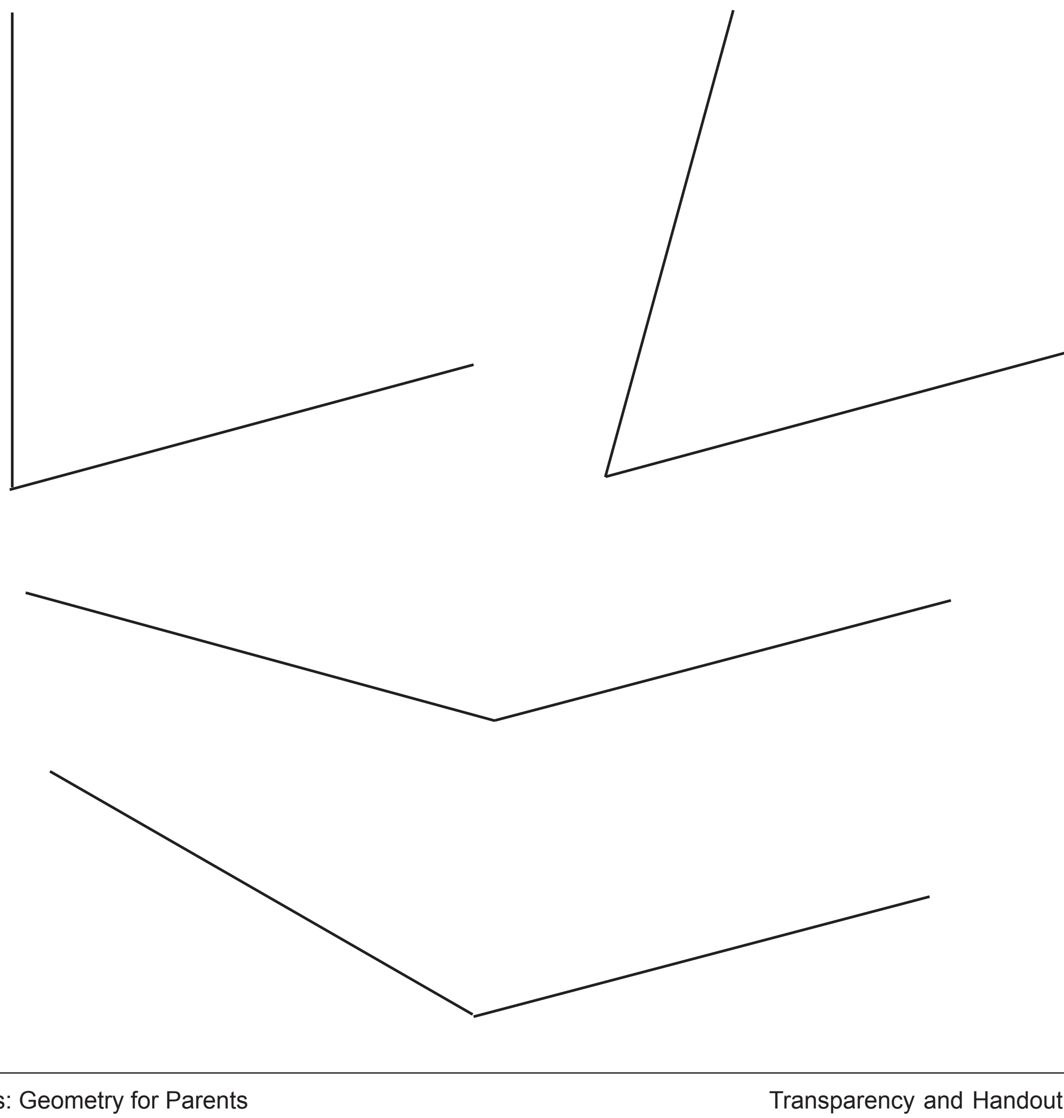
Geometry Triangles

Construction triangles are helpful to develop a feeling for the size of angles using benchmark angles 30°, 45°, 60° and 90°. Identify the sizes of the different angles in the geometry triangles.



Activity 2

Use the construction triangles to measure the size of the following angles.



Benchmarks for Angles

Activity 3

Use the construction triangles to estimate the size of the following angles. Describe the size of the angle using benchmarks as references; for example, the first angle is a little bigger than 30°, but smaller than 45°.

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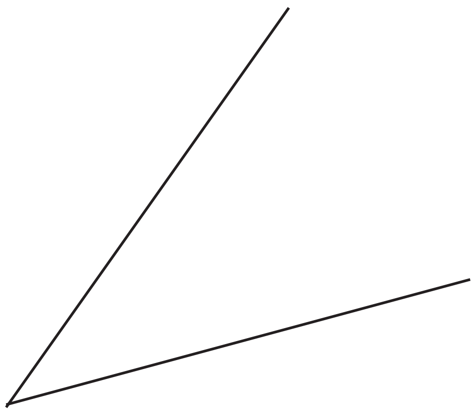

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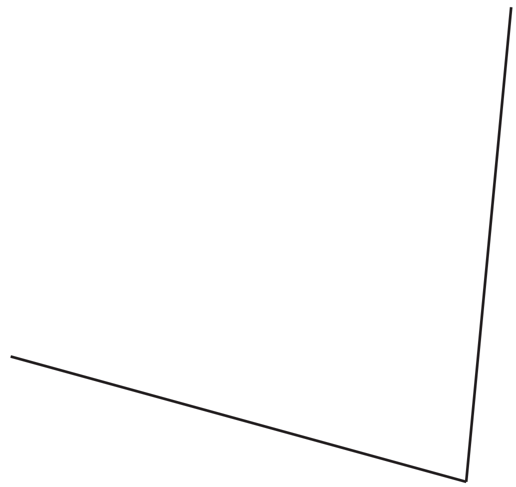

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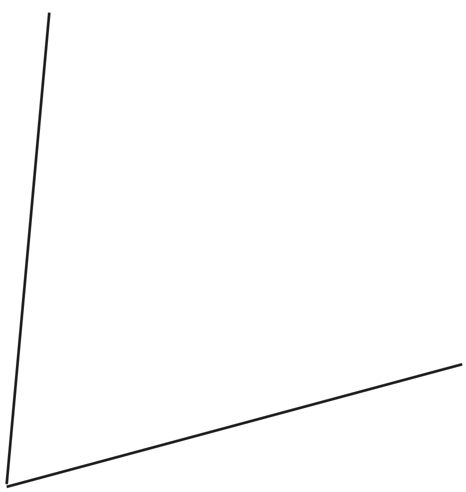

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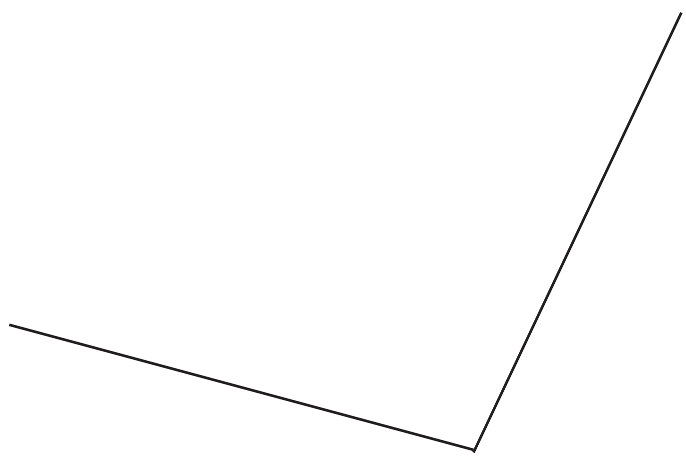
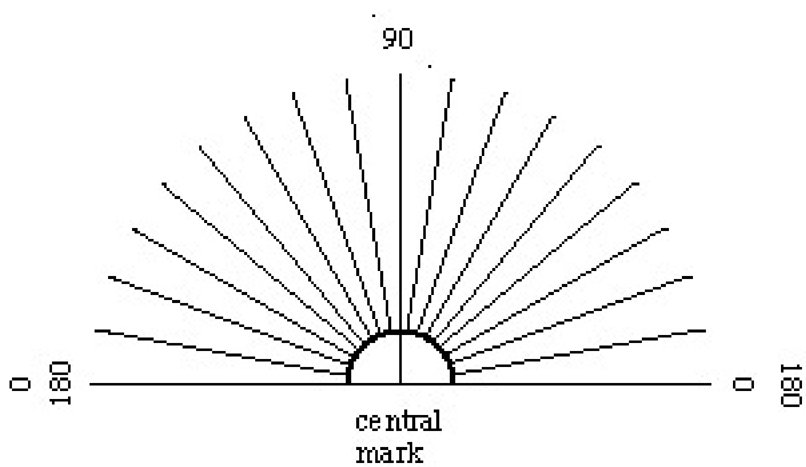


Measuring Angles with a Protractor

After measuring angles with wedges, and simple protractors (like the 15° protractor), you will understand better how to use the standard protractor. This instrument to measure angles is difficult for students for many reasons. The unit of measurement 1° is very small, so that we do not have an intuitive feeling of how big 1° is. Because 1° is so small, the subdivisions are very close together. Even if the marks are numbered only every 10°, the protractor looks very busy. There are two scales running simultaneously, one from left to right and a second from right to left. Some protractors do not have lines connecting the endpoints of the angle with its vertex, so that some people find it hard to understand what it is that they are measuring.

Activity 1. Three important points on the protractor

Find the marks corresponding to 0° and 180° on both sides of the protractor. There should be a line segment on the protractor joining the marks on opposite sides of the protractor. Find the mark for the midpoint of this segment (this we will call the central mark).



The central mark of the protractor needs to be on the vertex of the angle you want to measure. One leg of the angle should pass through the 0° mark on one side of the protractor. Look for the number closest to the other leg of the angle. That number will indicate the measure of the angle.

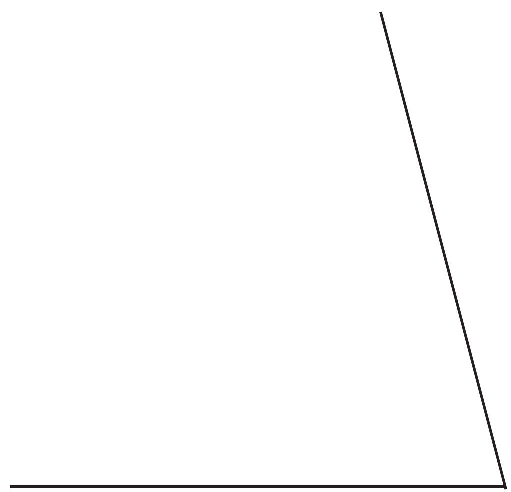
Measuring Angles with a Protractor

Activity 2. Measuring angles with the scale that goes left to right

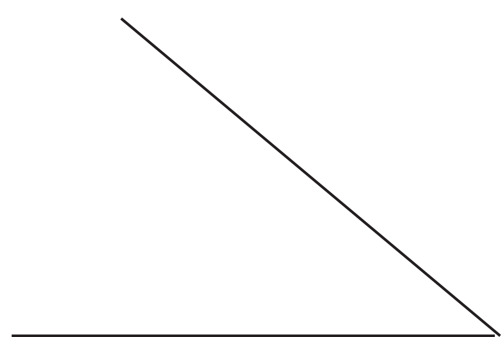
The following angles are easier to measure if we make the 0° mark on the left of the protractor coincide with the horizontal leg of the angle. Notice the scale going from 0° on the left to 90° on the top, and then continuing to 180° on the right. If you are measuring an angle smaller than a right angle, the measure should be less than 90° (angles less than a right angle are called acute angles). If you are measuring an angle bigger than a right angle, then you should get a measure bigger than 90° (these angles are called obtuse angles).

1.

2.



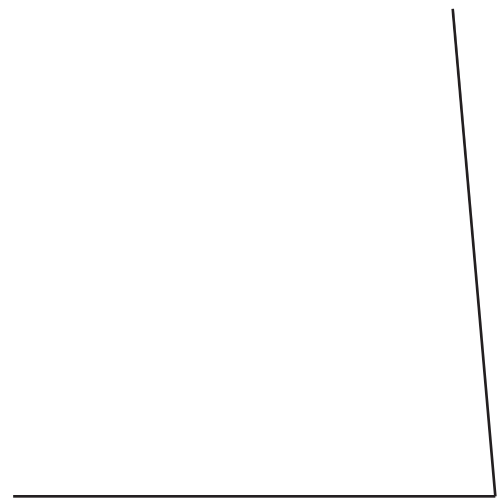
3.



Measuring Angles with a Protractor

Activity 2. Measuring angles with the scale that goes left to right (continued)

4.



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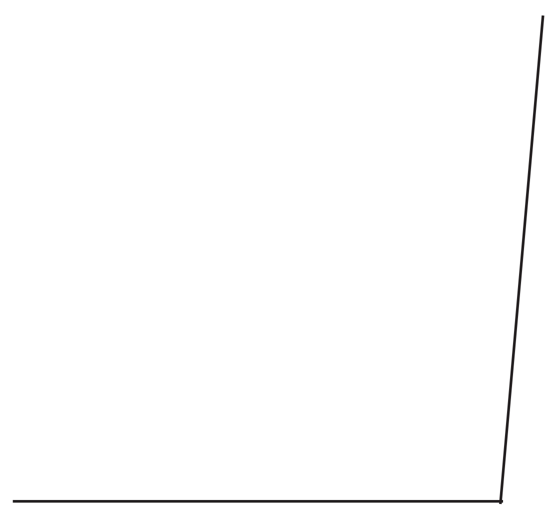

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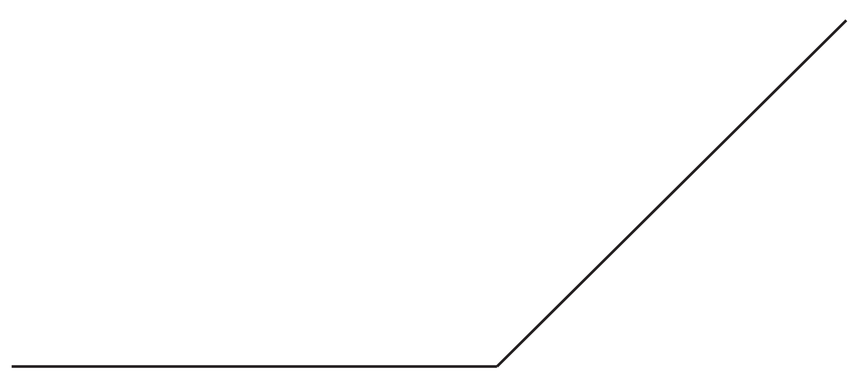
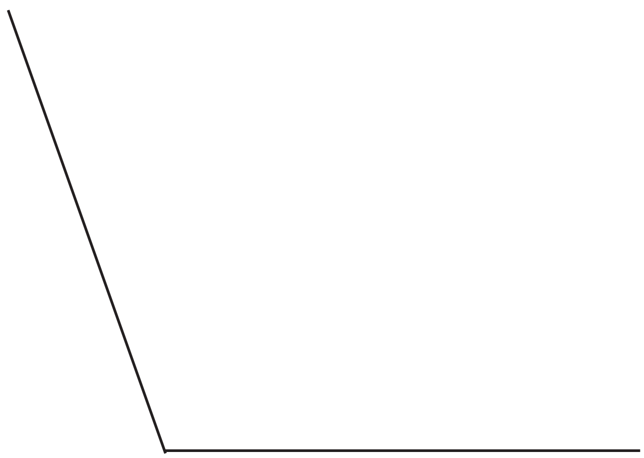


Measuring Angles with a Protractor

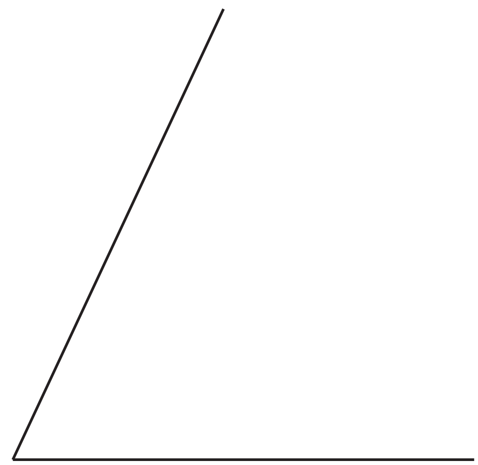
Activity 3. Measuring angles with the scale that goes right to left

The following angles are easier to measure if we make the 0° mark on the right of the protractor coincide with the horizontal leg of the angle. Notice the scale going from 0° on the right to 90° on the top, and then continuing to 180° on the left. If you are measuring an angle smaller than a right angle, the measure should be less than 90°. If you are measuring an angle bigger than a right angle, then you should get a measure bigger than 90°.

1.



2.



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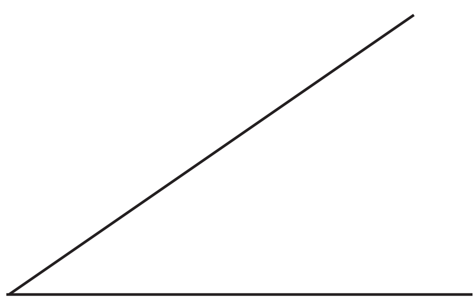

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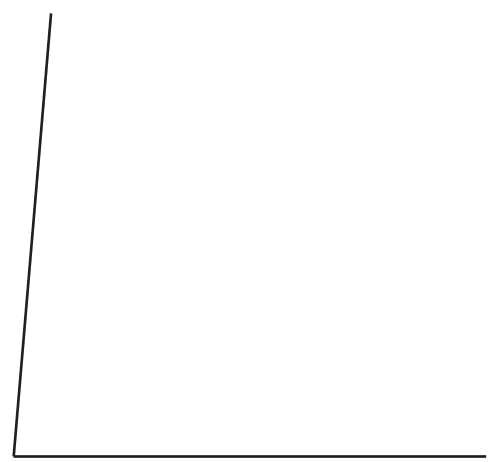

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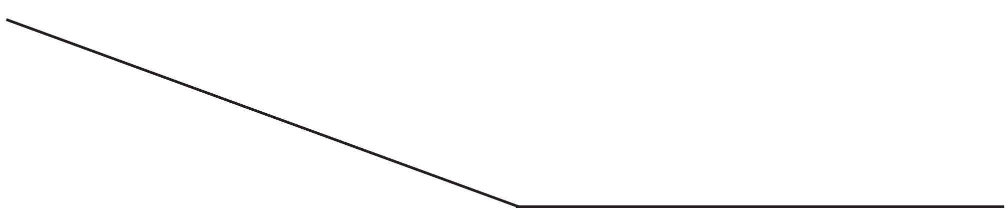
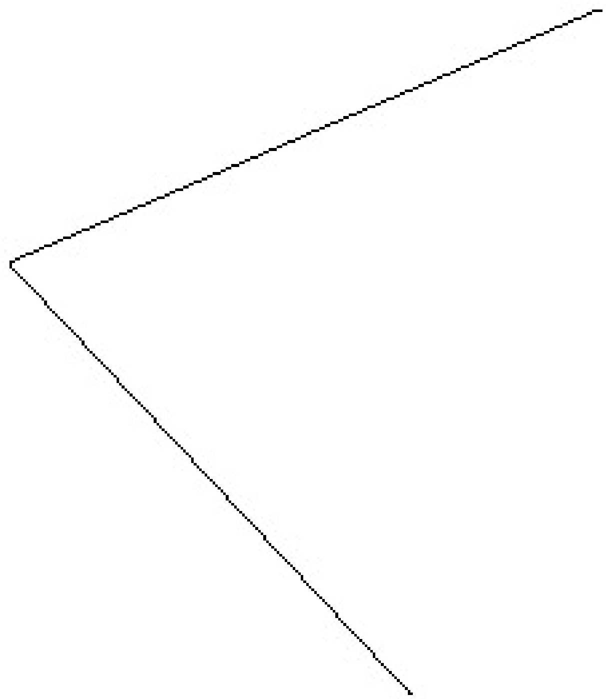


Measuring Angles with a Protractor

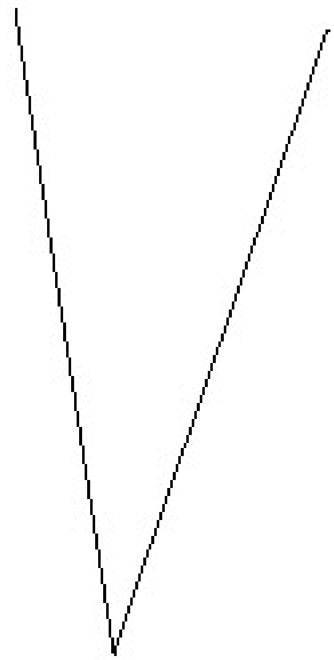
Activity 4. Using either scale

In many cases you will decide which scale makes it easier for you to measure the angle. Turn the protractor to make its base line coincide with one of the legs of the angle so that the central mark of the protractor is at the vertex of the angle.

1.



2.



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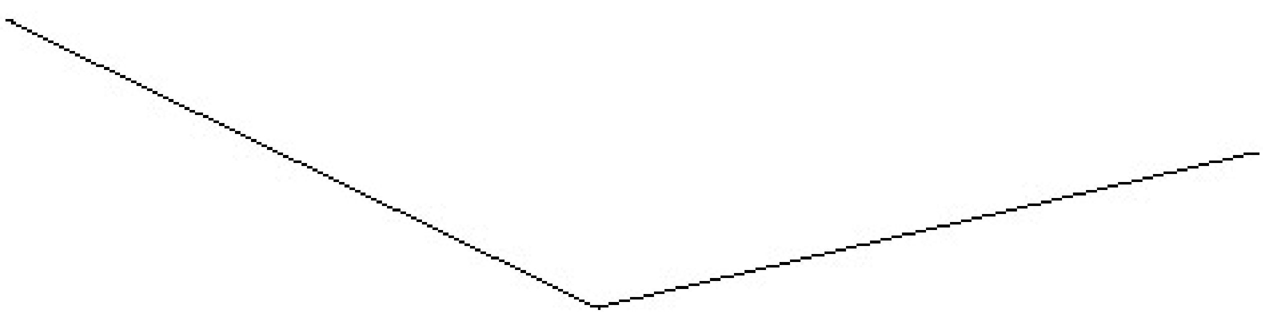


4.

Sum of the Angles of a Triangle

In these activities we will explore the sum of the angles of a triangle.

Activity 1. Tear and Join

1. Mark the angles of the paper triangle with a letter, or using different colors, or different marks. Tear the triangle with your hands (don't cut it with scissors) in three parts in such a way that each part has one of the angles.

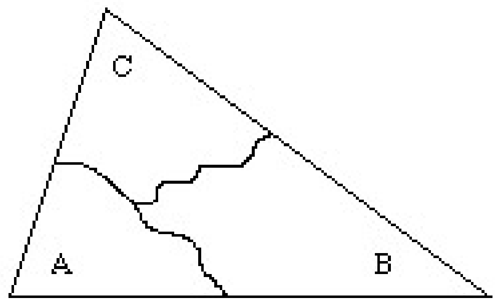


Figure 1. The triangle is torn in three parts

2. Reassemble the three parts so that the three angles are adjacent to each other and their vertices coincide in a point.

- *What do you observe?*
- *Look around you and observe what other participants obtained.*
- *You will observe that the three angles together form a straight line, as shown in figure 2.*

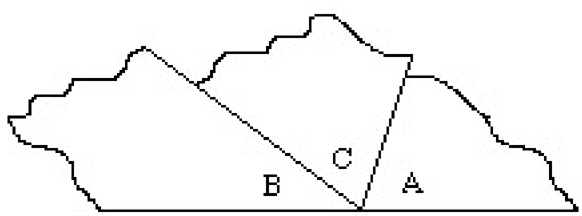


Figure 2. The three angles together

3. Remember that a straight line can be formed by two right angles, so the angle measure of a straight line is 180º. This experiment shows that the sum of the angles of your triangle is 180˚, and also for the triangles of the other participants.

Sum of the Angles of a Triangle

Activity 2. Verification by Measuring

We can also measure the angles of a triangle and obtain their sum.

1. Identify the inner angles of each of the triangles in Figure 3 on the next page.
2. Using the protractor, measure the inner angles of each of the triangles.
3. Write your results on the table.

	MEASURES OF THE ANGLES			SUM OF MEASURES
	A	B	C	A + B + C
1				
2				
3				
4				
5				
6				

4. Observe the column of the sum of the angles.
   They are close or equal to what value?

One objective of this activity is to convince you in a different way that the sum of the angles in a triangle is 180º. Another objective is that you see that when we measure there is usually a slight measuring error, so that not always do the values add exactly to 180°; the sum may be off by a few degrees.

Sum of the Angles of a Triangle

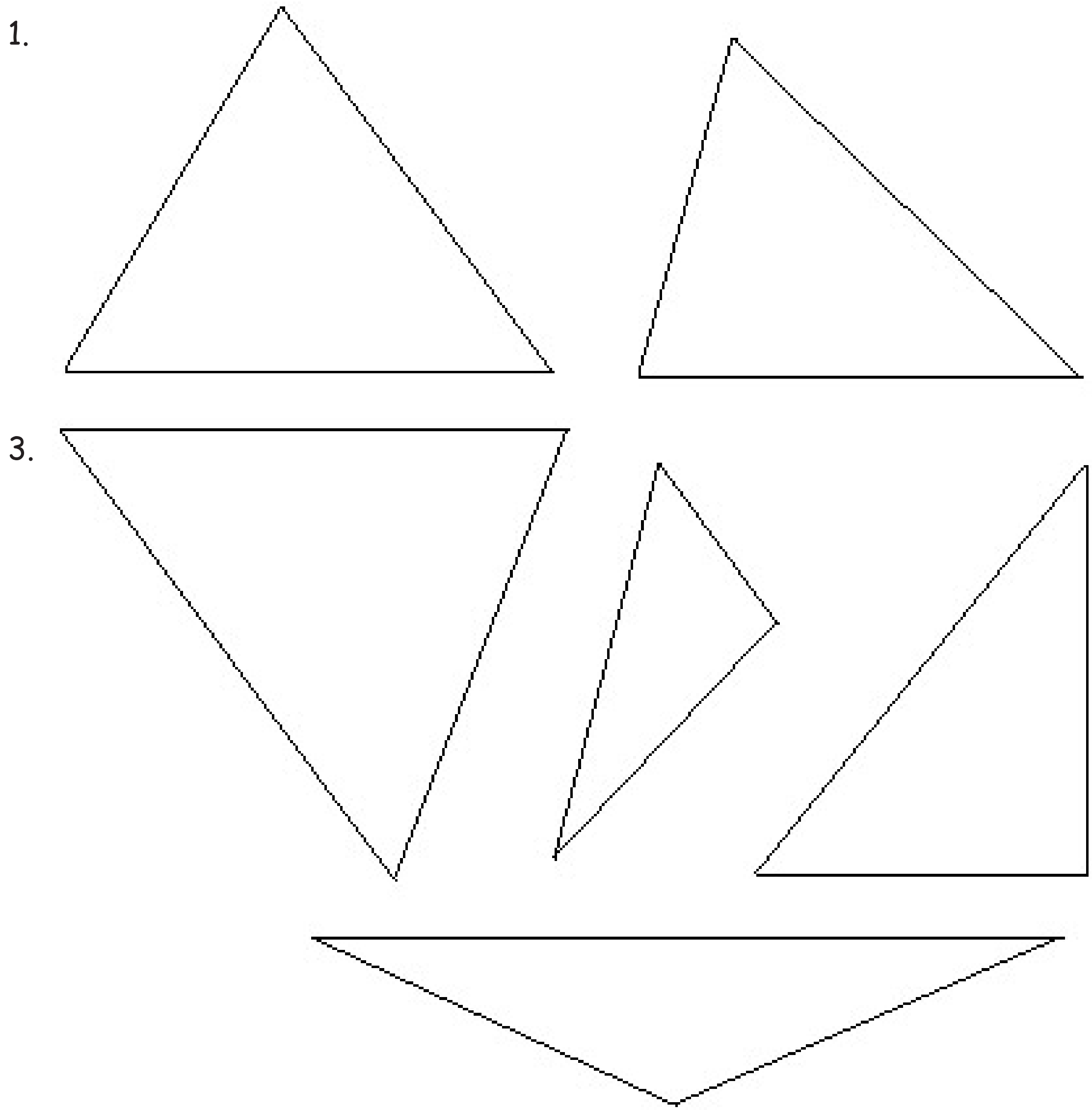


Figure 3. Triangles

triangle bigger or smaller but keep the same shape?

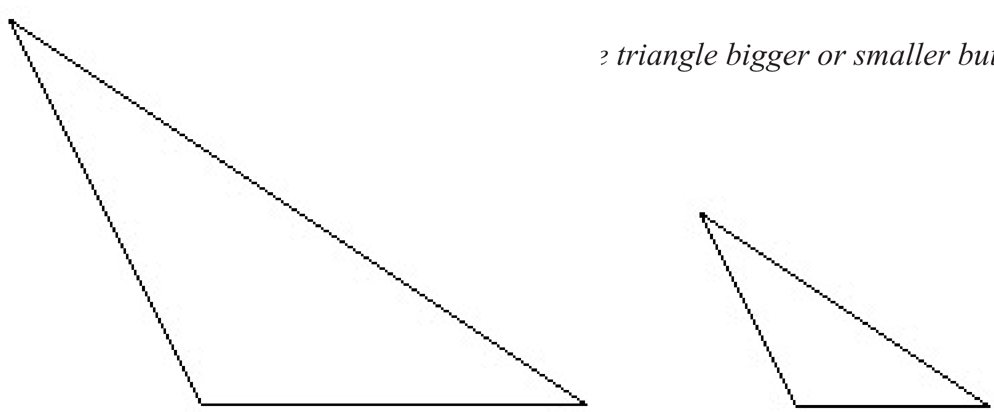


Figure 4. Angles in similar triangles

Sum of the Angles of a Triangle

Activity 3. Triangular Tiles

Now we will use triangular tiles for another method to arrive at the same result.

1) Paste the triangular tiles grid given at the end of this section on cardboard and cut out the triangles.

2) Observe that all the triangles are congruent to each other. Corresponding sides have the same length, and corresponding angles are equal. Observe also that angles that are equal have been marked with the same color.

3) Use the triangular tiles to form a straight path, with edges parallel to each other. There are different ways to form a straight path, here we show one in Figure 5.

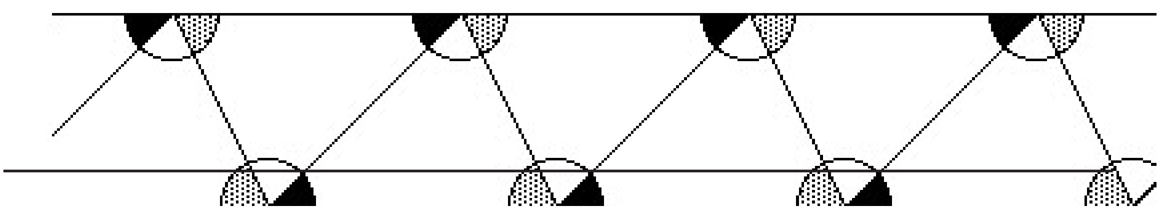


Figure 5. A straight path formed by congruent triangles.

4) Look at the angles that are together at one of the sides of the straight path. Notice that you have one of each color.

What is the sum of these three angles?

5) Now look at one of the triangles. See Figure 6. You will notice that it has angles that are equal to the corresponding angles that form the straight line.

What can you say about the sum of the angles in the triangle compared to the sum of the three angles at the side of the path that form the straight line?

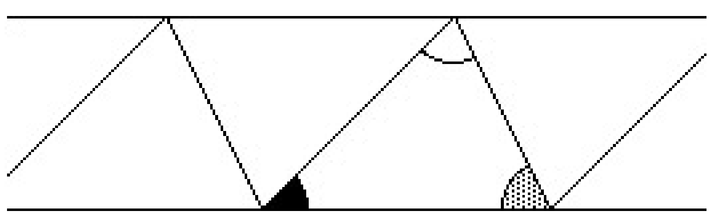


Figure 6. Three angles.

Sum of the Angles of a Triangle

Activity 3. Triangular Tiles (continued)

6) Use the triangular tiles to form a double straight path as in Figure 7.

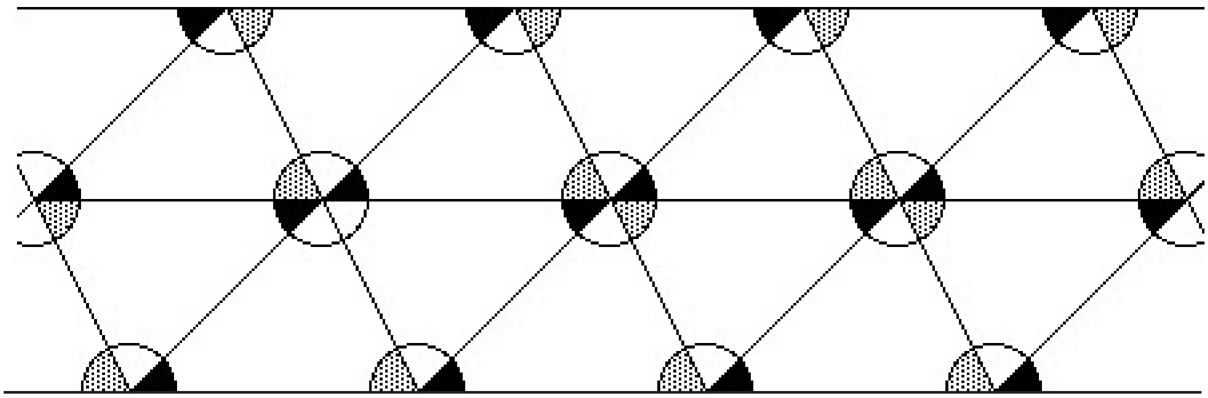


Figure 7. A double path

7) Look at one of the vertices in the center of the path in Figure 8. Notice that six angles surround one point.

What is the sum of the six angles?

8) Describe the pattern of colors of the angles.

- *How does that relate to the angles of one of the triangles?*
- *What can you say about the sum of the angles in a triangle?*
- *What can you say about angles that are opposite to each other?*

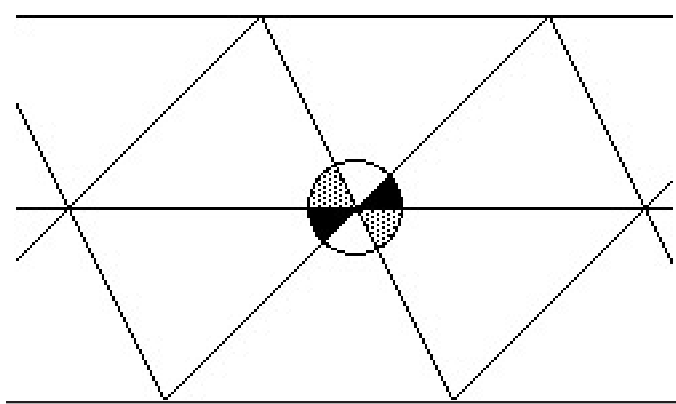
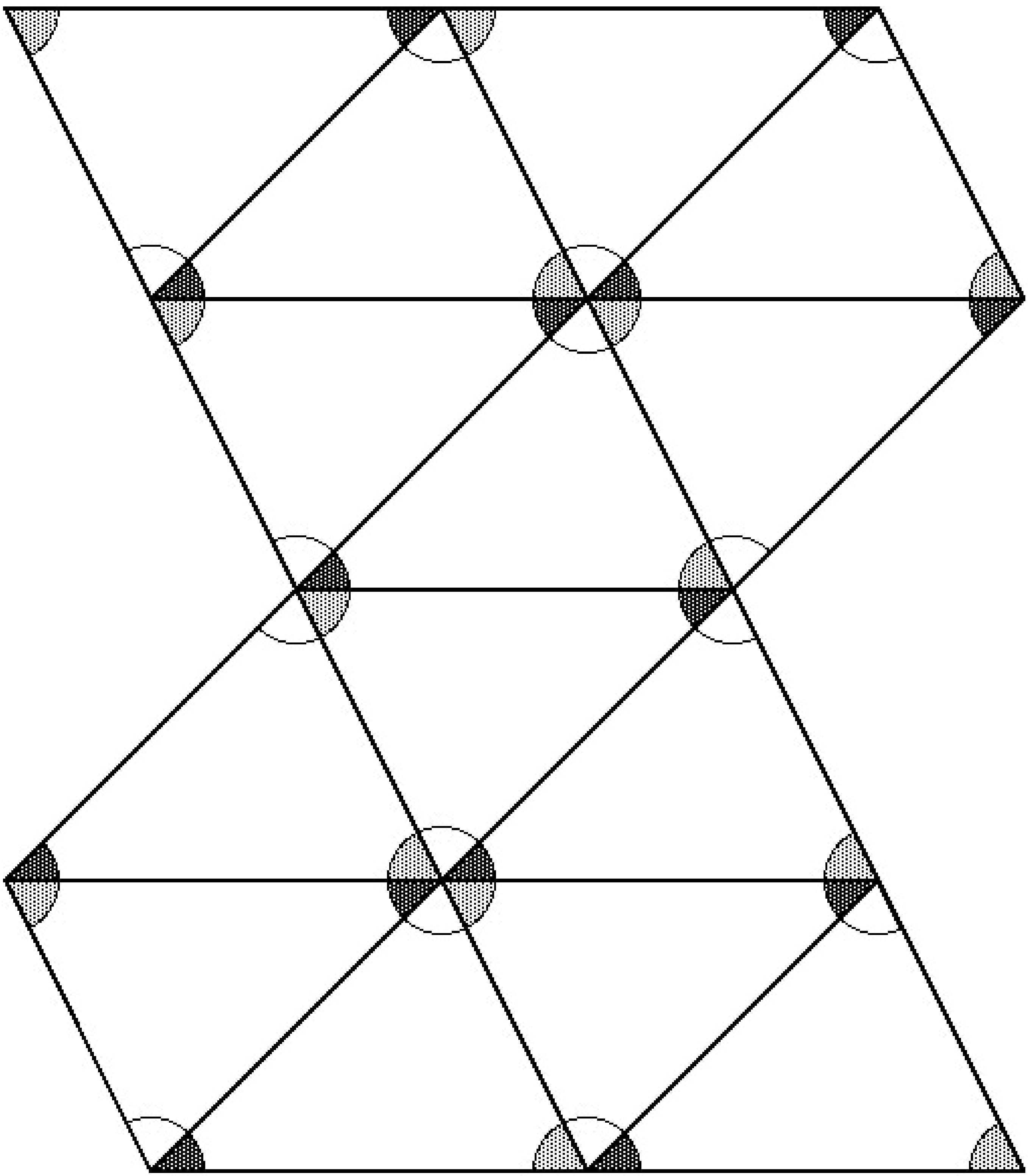


Figure 8. Six angles around a central vertex

Sum of the Angles of a Triangle

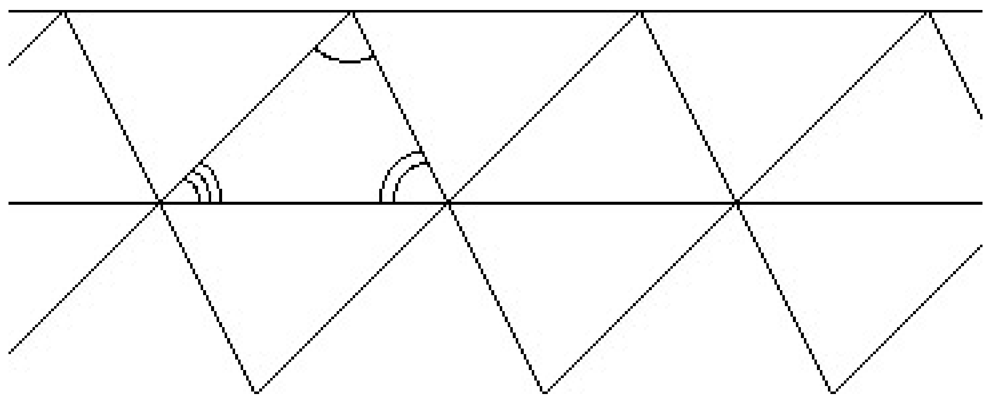
Paste on cardboard or copy on cardstock and cut out the triangles.



Additional Activity for Sum of Angles in a Triangle

Supplementary material for groups that move at a faster pace than other groups in the class.

Identify the angles that are equal to each of the angles marked in the figure, painting equal angles with the same color.



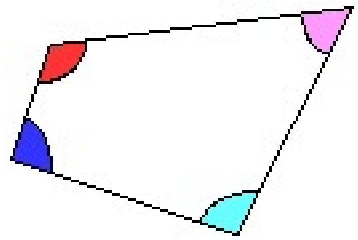
Identify equal angles

In the six angles that surround one vertex, observe the colors that correspond to the angles of the triangle. There are three pairs of colors; each color corresponds to an inner angle of the triangle. Use this result to obtain in another way the sum of the angles of the triangle.

The Sum of the Angles in a Quadrilateral

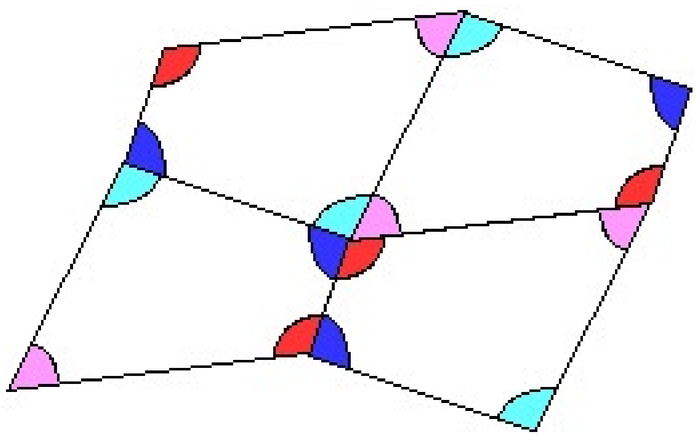
Activity 1

In this activity we will study the sum of the angles in an arbitrary quadrilateral.
You have a set of four congruent copies of an arbitrary quadrilateral, that is, it is not special in any way. Mark the angles in one of the quadrilaterals with different colors. Identify angles in the other quadrilaterals that are congruent to each of the colored angles. Mark congruent angles with the same color.



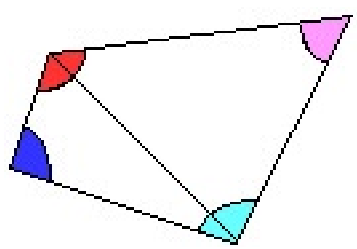
Identify angles in the other quadrilaterals that are congruent to each of the colored angles. Mark congruent angles with the same color.

A way to see that the sum of the angles in an arbitrary quadrilateral is by placing four copies of the quadrilateral around a common vertex, so that all colors are present. You will need to rotate two of the quadrilaterals.



Activity 2

Another method to find the sum of the angles in a quadrilateral is by realizing that the quadrilateral can be decomposed into two triangles.

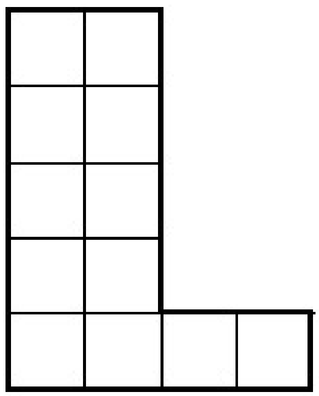


Use the fact that the angles of the two triangles together form the angles of the quadrilateral. And that the sum of the angles in each of the triangles is 180° to find the sum of the angles in the quadrilateral.

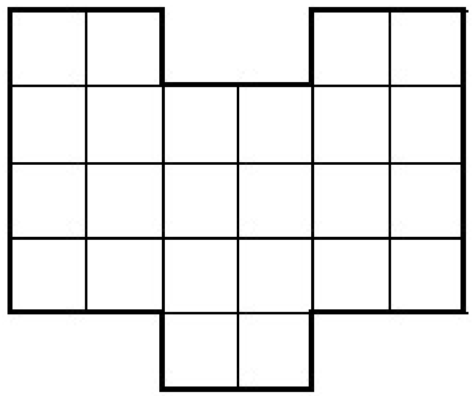
Area and Perimeter

Opening Activity

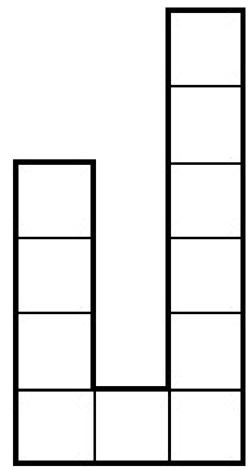
1) Find the area and the perimeter of the following shapes (the grid used is cm^2).
2) Describe what you did to find the area of each of the figures.
3) Describe in your own words what area means and what perimeter means.
4) Give examples of when area is used, and when perimeter is used.



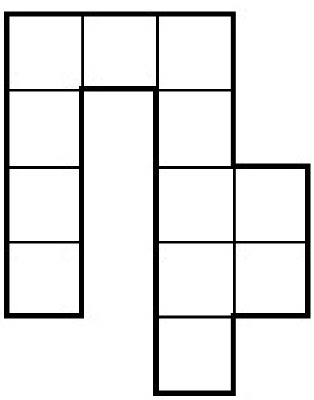
A =
P =



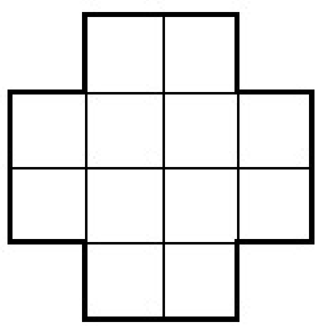
A =
P =



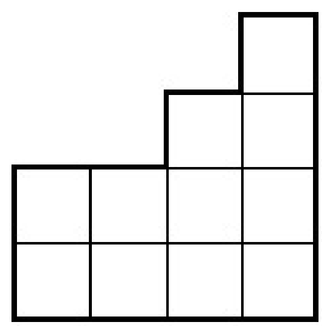
A =
P =



A =
P =



A =
P =



A =
P =

5) In your small group, discuss what area is and give a definition.
6) Discuss and define what is perimeter.

Centimeter Grid Paper

Area on the Geoboard

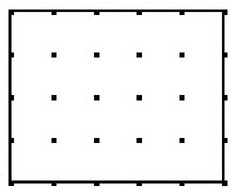
The geoboard is a board that has pegs forming a square grid. They are commercially available (for about $6). You can form geometric shapes using rubber bands. For these activities the square that is between four adjacent pegs on the geoboard will be considered of area 1 unit square. You can also use a grid and a ruler to trace the figures.

Activity 1

Construct one unit square on the geoboard. Construct a rectangle that contains the unit square. *How many unit squares could you fit in your rectangle?*

Activity 2

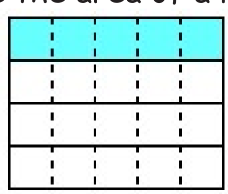
1) Construct a rectangle on the geoboard that has a base of five units and a height of four units.



Count the number of unit squares contained in the rectangle.

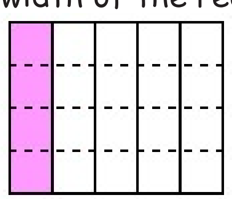
2) Use rubber bands to divide the rectangle into rows.
- *How many rows do you have?*
- *How many unit squares in each row?*
- *How can you use this to find the total numbers of unit squares in the rectangle?*

Notice that the number of squares per row is given by the length of the base of the rectangle, and that the number of rows is given by its width. Relate this to the formula length x width that is used to compute the area of a rectangle.



3) Divide the five by four rectangle into columns.
- *How many columns?*
- *How many squares in each column?*
- *Describe an alternative way to find the total number of squares in the rectangle.*

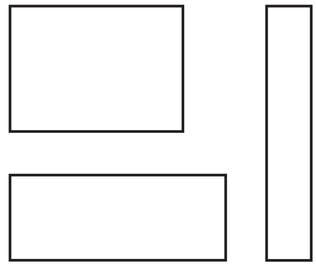
Notice that the number of columns is given by the length of the base, and the number of squares in each column is given by the width of the rectangle



Area on the Geoboard

4) Explain in your own words how the methods in 2 and 3 relate to the usual formula to compute the area of a rectangle: Area = length x width.

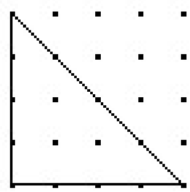
Construct a different rectangle. Find the total number of squares by counting, using rows, using columns, and by using a the formula length x width.



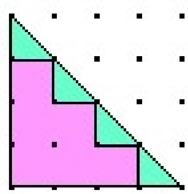
Activity 3. The area of a right triangle

A right triangle is one that has a right angle (90˚). Construct a right triangle on the geoboard that has its base parallel to the border, with a base of four units and a height of four units.

1) Find the area of the triangle by counting the number of unit squares contained within.



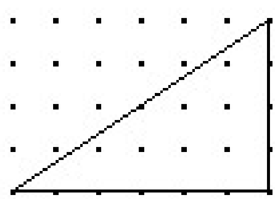
Here is one way to count the squares.



Construct a right triangle with a base of five units and a height of five units. Verify that there are 10 whole squares, and five half squares for a total area of 12-1/2 unit squares.

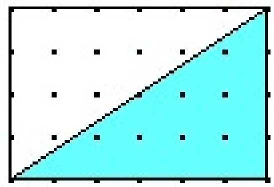
For other triangles it may not be as easy to count parts of squares. We will use a different method to find the area of a right triangle.

2) Construct a right triangle with a base of six units and a height of four units.

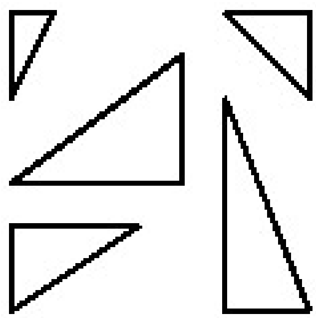


Area on the Geoboard

3) Show that the right triangle is half of a rectangle. *What is the total area of the rectangle?* Use this fact to find the area of the triangle. Notice that the base of the triangle is equal to the length of the rectangle, and the height of the triangle is equal to the width of the rectangle. If b is the base of the triangle, and h is its height, convince yourself that we can write the formula for the area of a right triangle as $\frac{b \times h}{2}$.



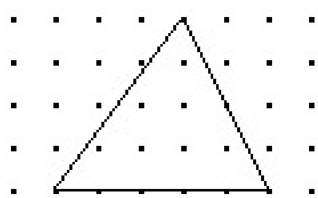
Construct a different right triangle. Find the total number of squares by showing that the triangle is half of a rectangle.



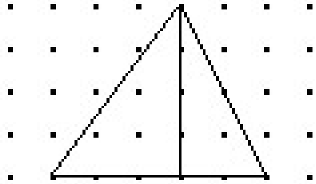
Activity 4. Areas of other triangles

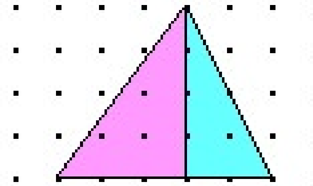
Construct a triangle that has its base parallel to one of the borders of the geoboard, and so that the angles at the base are acute (less than 90°).

1) Find the area of the triangle.



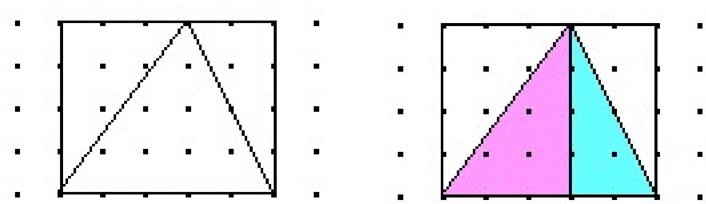
2) One method to find the area of the triangle is by building two right triangles as shown. We already know how to compute the area of a right triangle ($\frac{b \times h}{2}$). *How can you use this?*





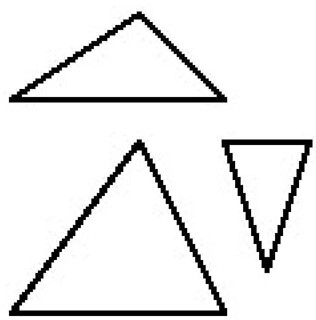
Area on the Geoboard

3) A second method is illustrated. Build a rectangle around the triangle with the same base and the same height. Use this method to find the area of the triangle.



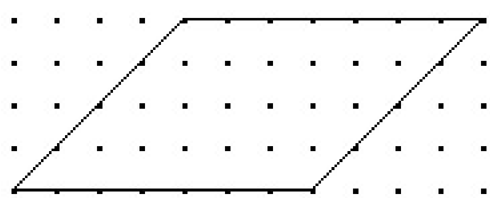
In all the cases we have seen here, the area of the triangle can be computed as base times height divided by two, the same formula for the right triangle. So, for the kind of triangles we have studied so far we can say Area = base × height / 2. We will see later that indeed the same formula can be used for any triangle.

Exercise. Construct a different triangle. Find the total number of squares by showing that the triangle is half of a rectangle.



Activity 5. The area of a parallelogram

1) Construct a parallelogram that has one base parallel to the border of the geoboard.



Find the area of the parallelogram.

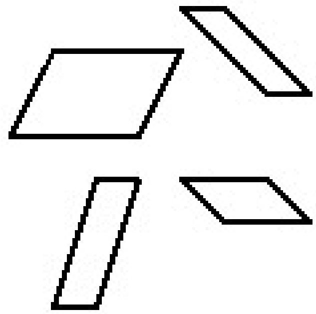
2) Construct your own parallelogram. Find a rectangle that has the same base (length) and the same area as the parallelogram.

- *How does the height of the parallelogram compare with the width (height) of the rectangle?*

Try to find the area of the parallelogram on your own, but if you can't, look at the hints and solutions at the end of this activity.

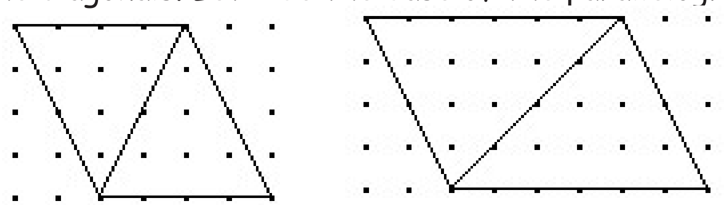
Area on the Geoboard

Explain in your own words why the area of the parallelogram is given by
Area = base x height



3) Construct a different parallelogram. Find a rectangle with the same area and the same base. Compute the area of the parallelogram.

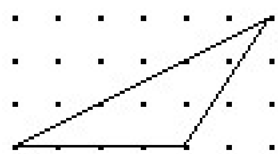
An alternative way to find the area of a parallelogram.
Construct a parallelogram. Divide the parallelogram into two congruent triangles by constructing one of the diagonals. Let *b* be the base of the parallelogram, *h* its height.



The area of each triangle is *1/2 x b x h. What is the area of the parallelogram?*

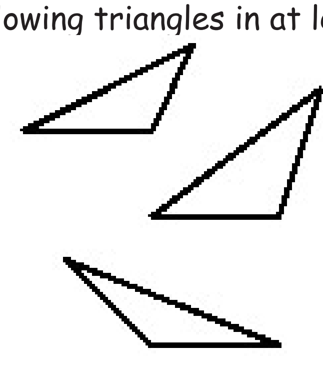
Activity 6: The area of a triangle

One kind of triangle that some children find more difficult is when the angle at the base is obtuse (bigger than 90°). Here we have a triangle with a base of 4, and height of 3, with an obtuse angle.



1) Find the area of this triangle. Try to find the area of the triangle on your own, but if you can't, look at the hints and solutions at the end of the handout.

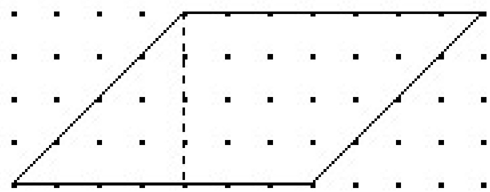
2) Find the area of one of the following triangles in at least two different ways.



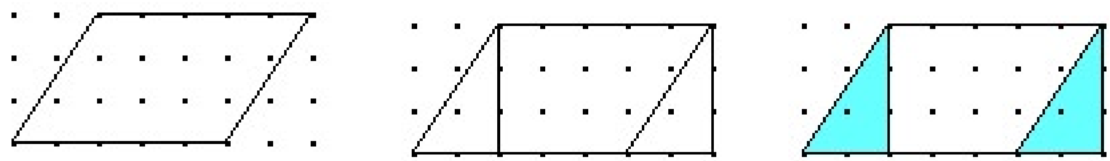
Hints and Solutions

Hints and Solutions for Activity 5: The area of a parallelogram

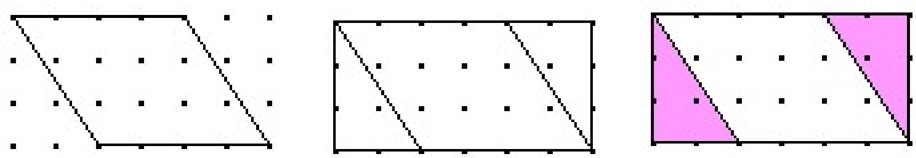
Here is one hint for a method you could use. Construct a perpendicular line to the base to form a triangle on one of the sides of the parallelogram.

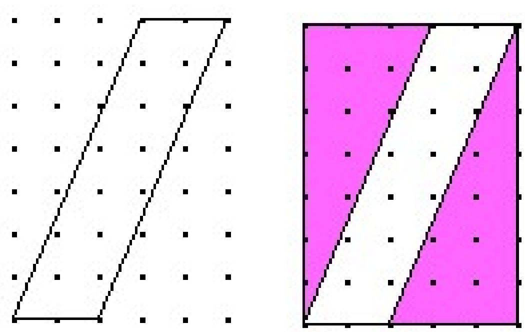


One possible solution



An alternative method to compute the area of a parallelogram is to form a rectangle that encompasses the parallelogram, compute the area of the whole rectangle, and subtract the areas of the two shaded triangles.

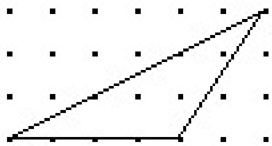
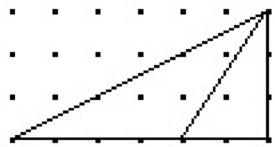
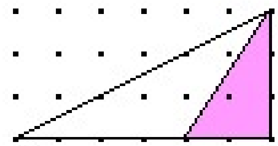




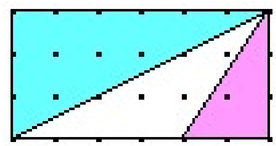
Hints and Solutions

Hints and Solutions for Activity 6: The area of a triangle

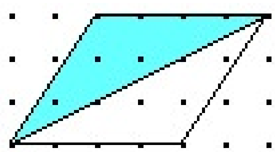
One way to find the area is by constructing a small right triangle, so that the original triangle together with this form a right triangle. The area of the original triangle will be the difference of the areas of the two right triangles.

A second method to compute the area of an obtuse triangle is to form a rectangle around it, compute the area of the rectangle and subtract the area of the blue triangle and the area of the purple triangle.



A third method is using the fact that the triangle is half a parallelogram with the same base and the same height. The area of a parallelogram is 4 x 3, the area of the triangle will be 1/2 x 4 x 3.



A Puzzle for the Area of a Parallelogram

Supplementary material for groups that move at a faster pace than other groups in the class

Paste the following pieces on cardboard and cut them out.

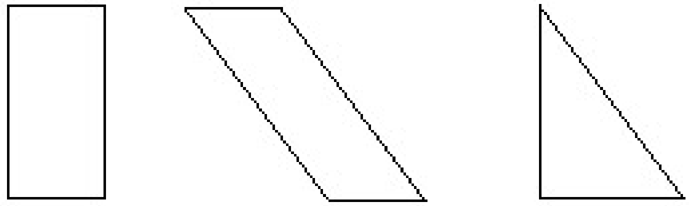


Figure 1

With the parallelogram and the triangle form the following puzzle.

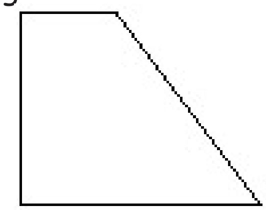


Figure 2

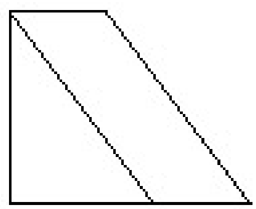


Figure 3

Now use the rectangle and the triangle to form the puzzle.

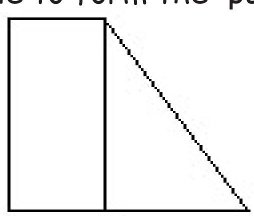


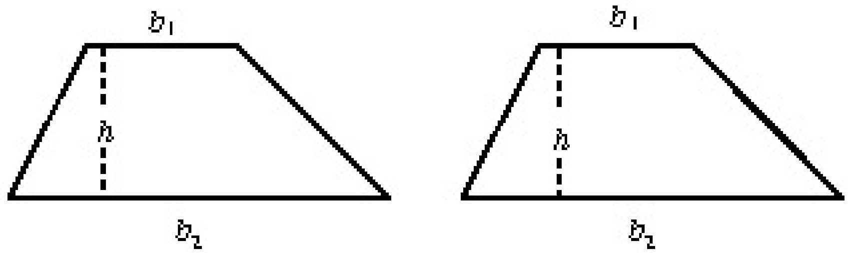
Figure 4

What can you say about the area of the parallelogram and the area of the rectangle?

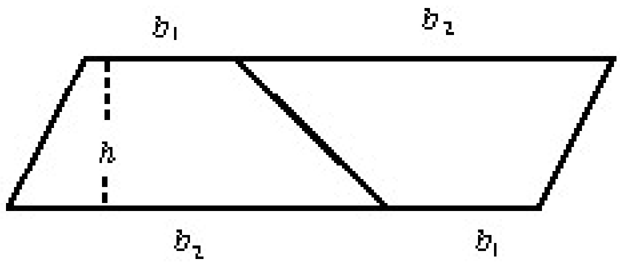
- The base of the parallelogram is equal to the base of the rectangle. You can see this by overlapping the two bases or by comparing Figure 3 with Figure 4.
- The height of the parallelogram is equal to the height of the rectangle. You can see this by comparing figures 3 and 4.
- What can you conclude about the area of a parallelogram with respect to the area of the rectangle with the same base and the same height?

The Area of a Trapezoid

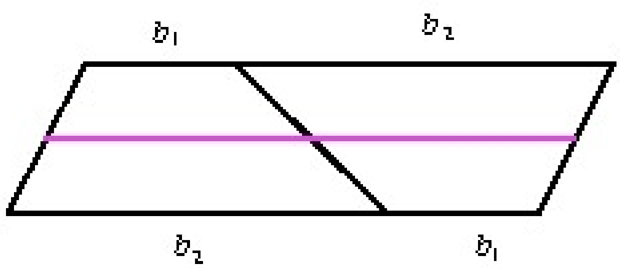
Cut two identical trapezoids. Let b_1 be the length of the shorter base of the trapezoid, b_2 the length of the other base, and h its height (see figure).

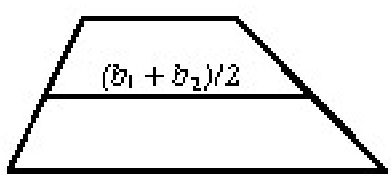


Turn one of the trapezoids so that both together form a parallelogram. *What is the length of the base of the parallelogram in terms of b_1 and b_2 ? What is its height? What is the area of the parallelogram? Relate the area of the trapezoid to the area of the parallelogram.* Write a formula for the area of the trapezoid.



Another interpretation for the formula.
For each trapezoid, draw the line that passes through the midpoints of the non parallel sides of the trapezoid. This line is called the median. Turn one trapezoid to form a parallelogram. *What do you observe about the two medians of the trapezoids?*

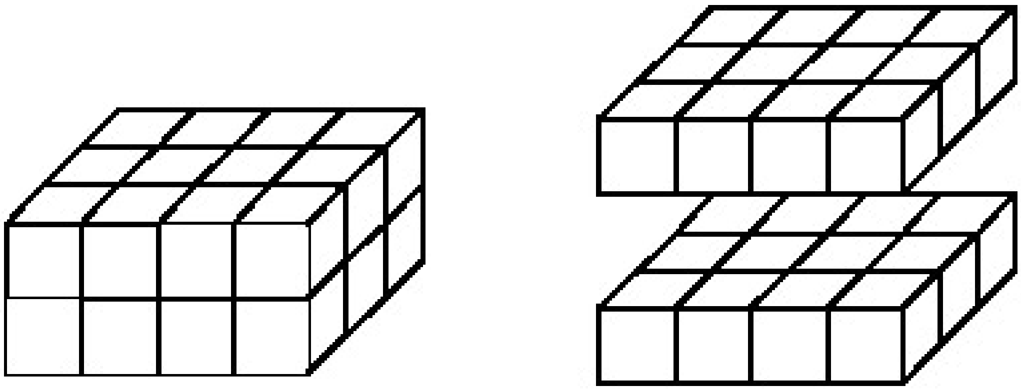




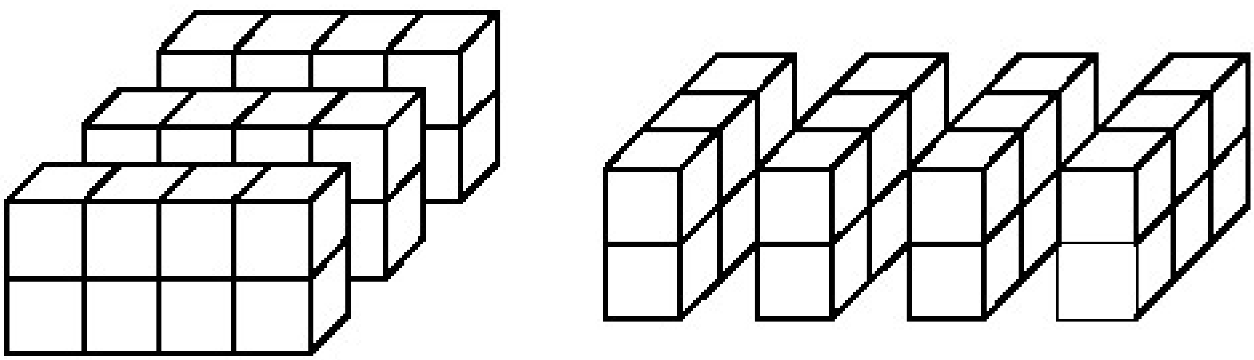
Give a convincing argument that the length of the median of the trapezoid has a length of $(b_1+b_2)/2$. Explain why another interpretation of the formula for the area of the trapezoid is to multiply the length of this median times the height.

Shapes

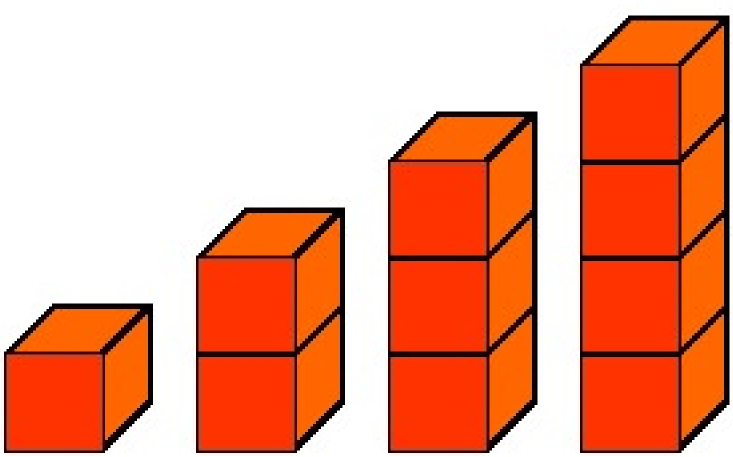
Activity 1. Explaining the formula for volume



Activity 3. Volume and Associativity of Multiplication



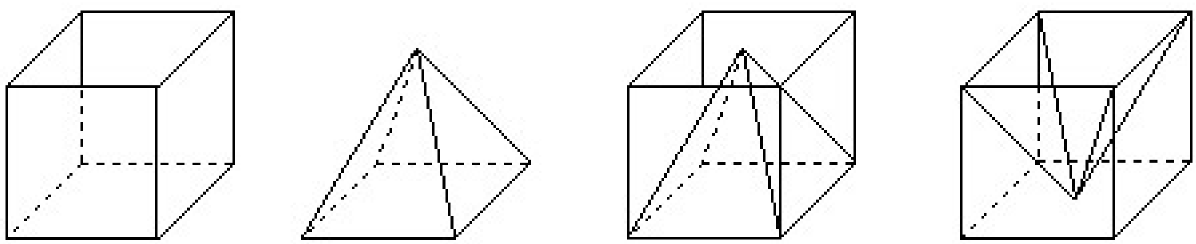
Activity 4. Increasing Towers Example



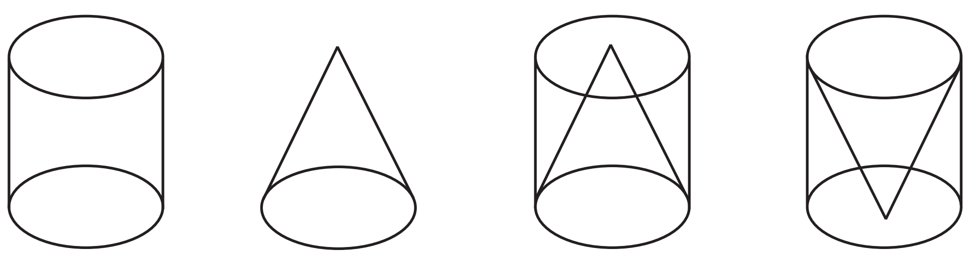
Number of Cubes (N)	Surface area in Square Units (S)
1	6
2	
3	
4	

Shapes

Activity 6. Volume of a Pyramid with Square Base



Activity 7. Volume of a Cone



Volume

Measuring a volume is compared with the unit volume (a 1 x 1 x 1 cube of volume 1).

Opening Activity

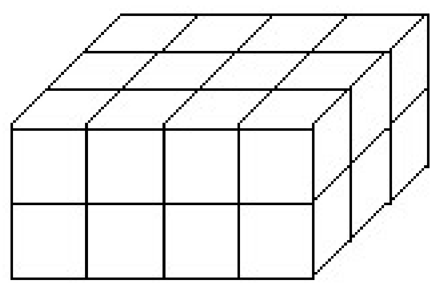
Build a two by three by four rectangular solid using the unit cubes. How many cubes did you use? How can you count the number of cubes in a systematic way?

The number of cubes in this rectangular solid can also be obtained by multiplying 2 x 3 x 4, that is, multiplying length times width times height. Discuss in your group why this is so. Break your rectangular solid into slices. Count the number of cubes in one slice. There are several ways to slice the rectangular solid. Depending on the slicing describe what do the partial products 2 x 3 or 2 x 4 or 3 x 4 represent.

Volume formulas

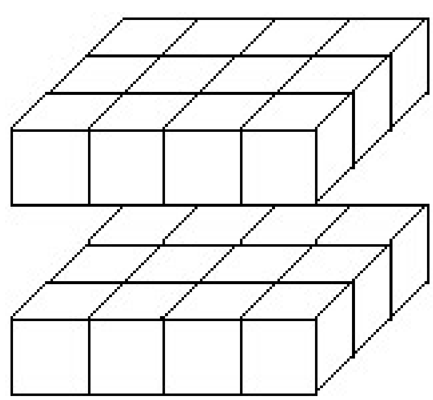
Activity 1. Explaining the formula for volume Base times height

1) A rectangular box has a base of area B and a height of length c. The volume of the box is V = area of the Base x height. Justify this formula in terms of the number of unit cubes inside the box.



2) Imagine the prism cut into slices (1 unit thickness). Compute the number of unit cubes in a slice.

- *How is this related to the area?*
- *How many slices?*
- *How is this related to the height?*



Activity 2. Explaining the formula length times width times height

1) A rectangular box has dimensions a, b, c. Another formula for the volume of the rectangular box is V = a x b x c. Justify this formula in terms of the number of unit cubes inside the rectangular box.

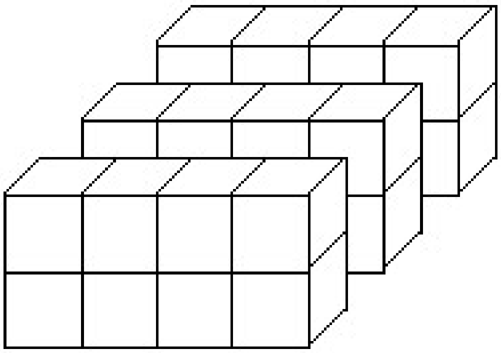
2) Show that the two formulas are equivalent. The volume of water for irrigation purposes is measured sometimes in acre-feet.

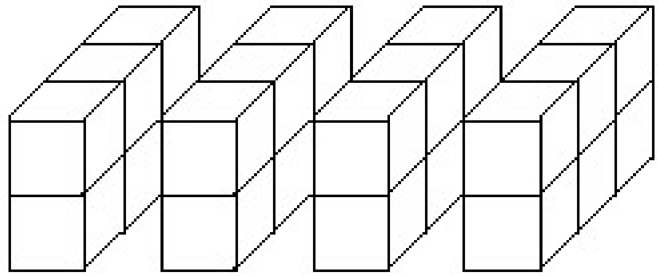
- *What formula is being used?*

Volume

Activity 3. Volume and associativity of multiplication

Multiplication is associative. That is, for any numbers a, b, c we have a × (b × c) = (a × b) × c. Describe how a brick can be sliced in two different ways to illustrate this property.



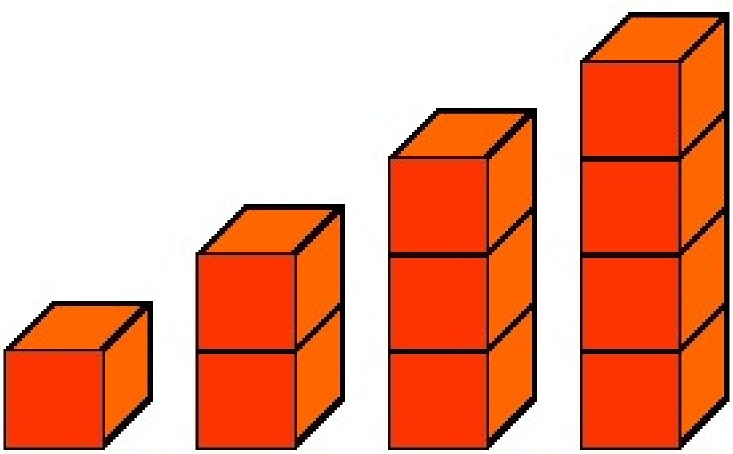


3 × (2 × 4) = (3 × 2) × 4

Activity 4. Increasing towers example

1) Build towers with wooden cubes.
   - *What is the surface area of each tower of cubes?* (not including the bottom)

2) Organize the information in a table, describe the relationship between surface area and the number of cubes in your own words, notice patterns, and describe the relation with an equation.



Number of Cubes (N)	Surface area in Square Units (S)
1	6
2	
3	
4	

3) Notice that the surface area increases by four when we add another cube to the tower (*why?*). Express the surface area as a function of the number of cubes, first in plain English, and then using symbols.

Activity 5. Surface and volume of similar figures

A very different situation in the way volume and surface area change is when all dimensions of a figure change in the same proportion. Compute the surface area and the volume of a 2 × 3 × 5 rectangular box. If each of the dimensions of the box is multiplied by 2, we obtain a new box, 4 × 6 × 10.

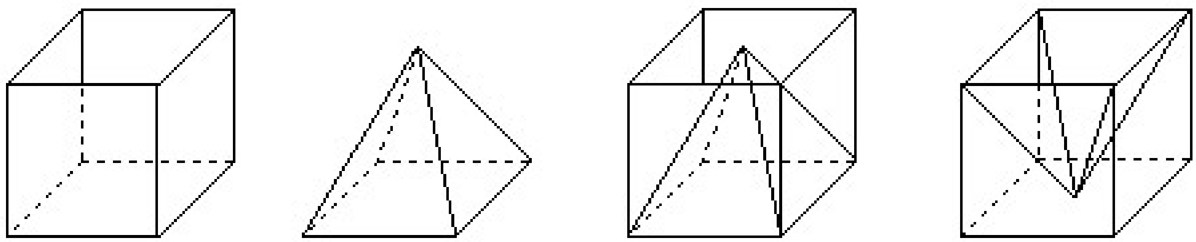
- *What is the surface area of the new box?*
- *What is its volume?*

Justify your answer in at least two different ways. Notice that the surface area increased by a factor of 4, whereas the volume increased by a factor of 8.

Volume of a Pyramid and Cone

Activity 6. Volume of a pyramid with square base

1) The following figures represent a cube and a pyramid with the same base and the same height. To determine the volume of the pyramid, we can fill it with beans (or water), and pour the content repeatedly into the cube.



- *How many times do you think the volume of the pyramid will fit inside the volume of the cube with the same base and the same height?*

Make a prediction, then conduct or observe the experiment.

2) Verify that the cube and the pyramid have the same height, and that the area of their bases is the same. Fill the pyramid with packing material or with seeds. Pour the content of the pyramid into the open cube. Repeat the procedure until the cube is filled. Count how many times does the volume of the pyramid fit into the cube.

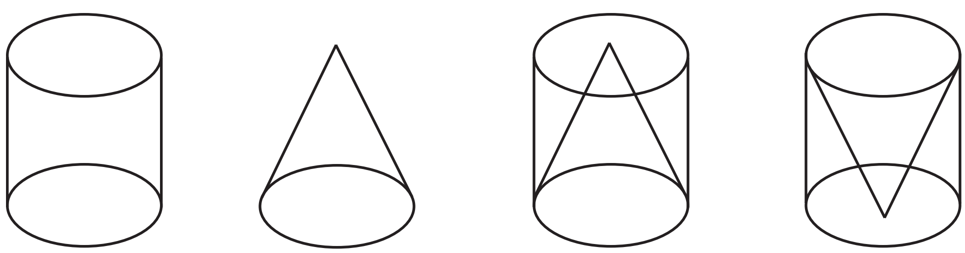
3) Let B be the area of the base, and h the height of the cube. The volume of the cube is given by $V = B \times h$.

- *Based on the results of the experiment, what would be a formula for the volume of the pyramid?*

Activity 7. Volume of a cone

1) The following figures represent a cylinder and a cone that have the same base and the same height.

- *What is the volume of the cone compared to the volume of the cylinder?*



- *How many times do you think the volume of the cone will fit inside the volume of the cylinder with the same base and the same height?*

Make a prediction, and then do the experiment.

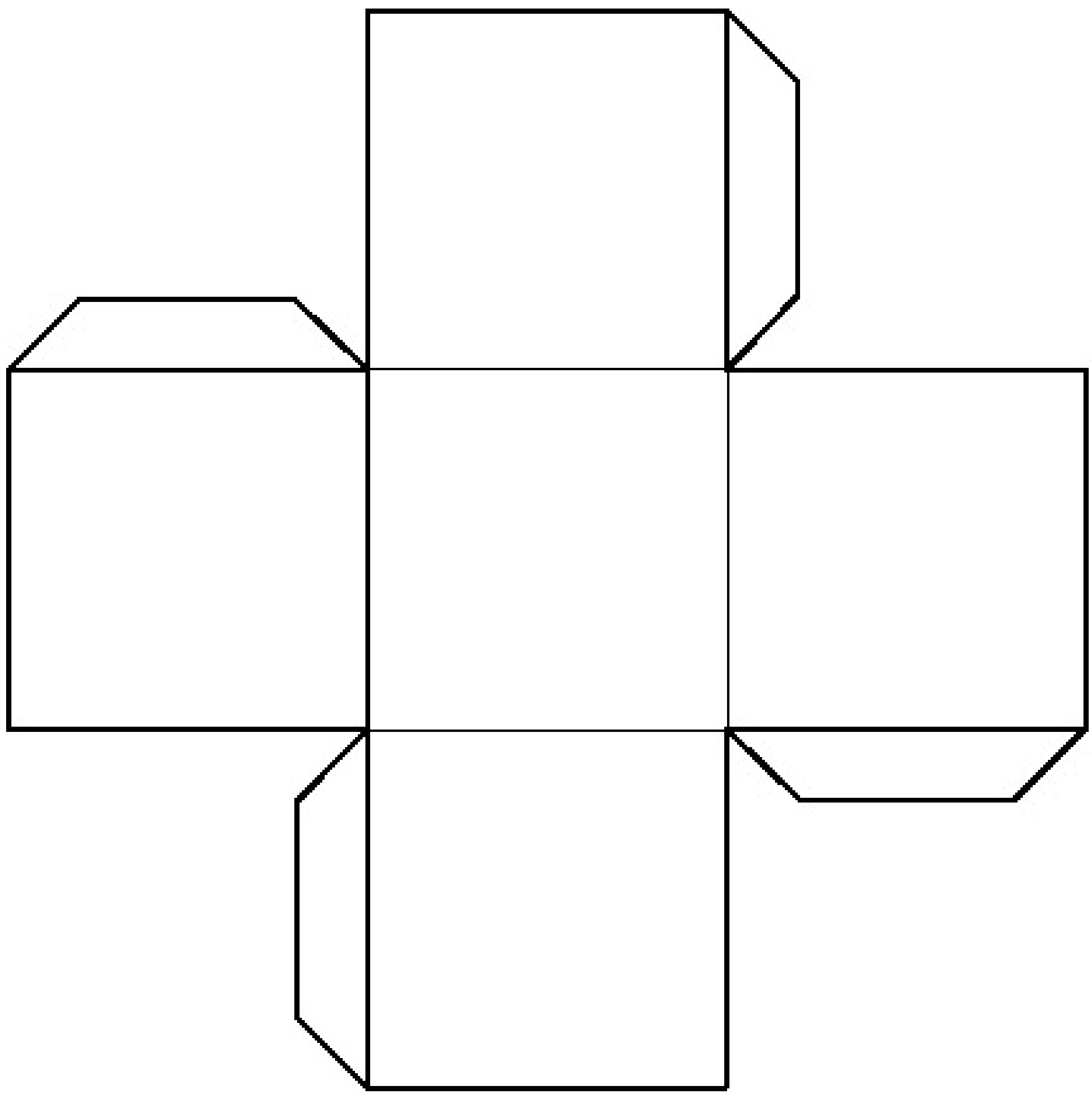
2) Fill the cone with beans or other small light objects. Pour the content of the cone into the cylinder. Repeat the process until the cylinder is filled. Count how many times does the volume of the cone fit into a cylinder with the same base and height.

3) The volume of the cylinder is given by the product of the area of its base times its height, $V = B \times h$.

- *Based on the experiment above, what would be a formula for the volume of a cone?*

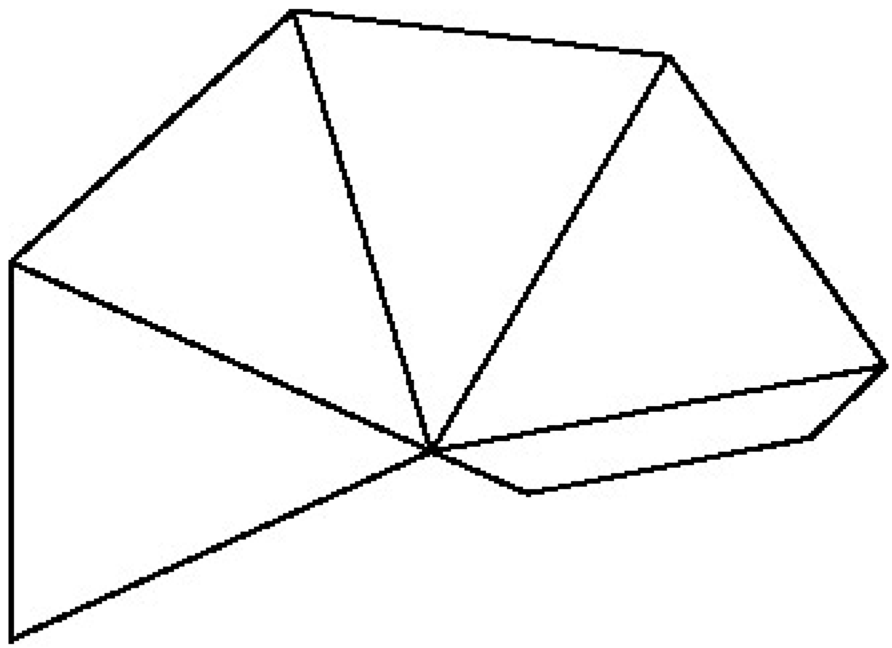
Cube

Paste on cardboard and cut. Paste the flaps to form an open cube.



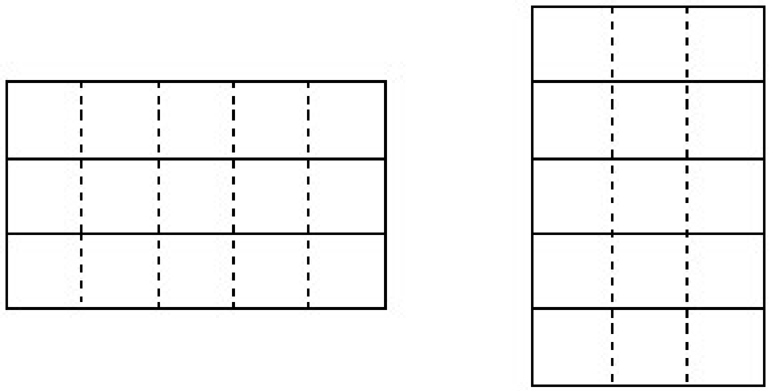
Pyamid

Paste on cardboard and cut. Paste the flap to form a pyramid without a base.



Geometric Representations of Arithmetical Operations

Use area model to illustrate commutative property of multiplication and provide your own examples. Describe the two rectangles first in plain English and then with numbers. "Three rows of five squares", "Five rows of three squares."



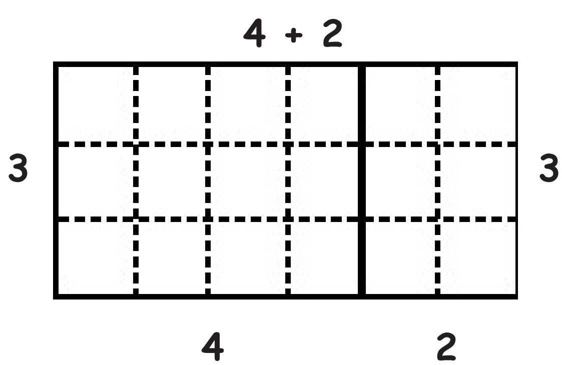
5 x 3 = 3 x 5

The use of geometric diagrams can help students see algebraic properties at a glance. In this case, the area model shows immediately that multiplication is commutative ("just rotate the rectangle"). Unless the compute the sum on both sides, this may not be evident to students from the representation of multiplication as repeated addition.

3 + 3 + 3 + 3 + 3 = 5 + 5 + 5

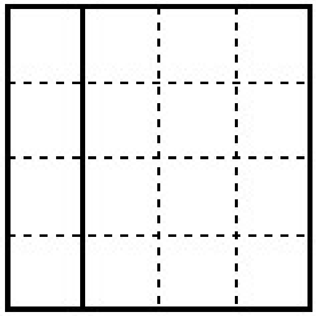
Geometric representation of distributive property

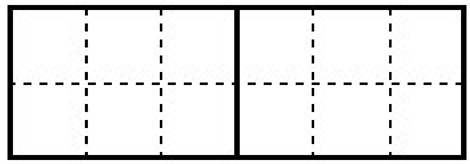
The following rectangle illustrates (4 + 2) × 3 = 4 × 3 + 2 × 3

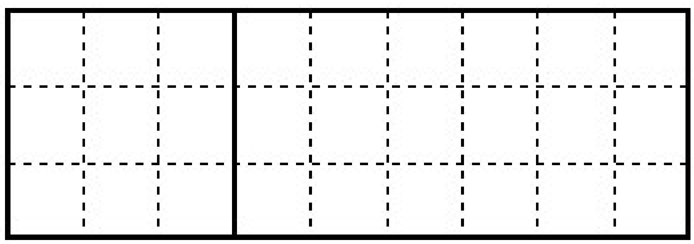


Geometric Representations of Arithmetical Operations

Write the corresponding distributive equalities for the following rectangles.







Use tiles or a grid to illustrate the following identities:

$$(5 + 3) \times 3 = 5 \times 3 + 3 \times 3$$

$$(10 + 2) \times 3 = 10 \times 3 + 2 \times 3$$

Geometric Representations of Arithmetical Operations

Making distributive property explicit in multiplication of two-digit numbers

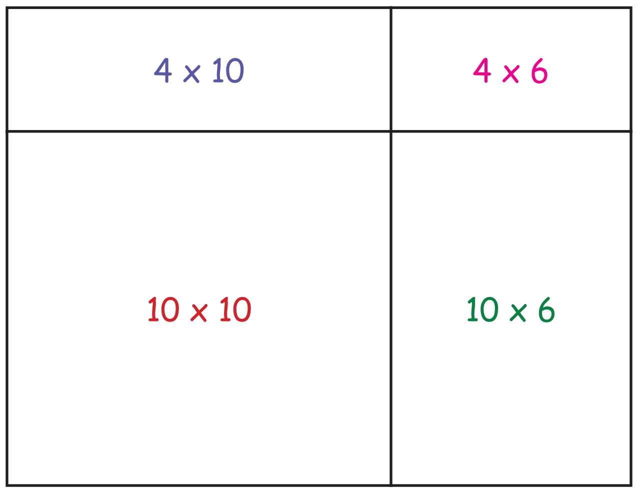
Discuss an explicit procedure for multiplying 14 x 16; relate it to the algorithm you know, and to a geometric representation.

16	14
x 14	x 16
64	84
160	140
224	224

For the first procedure, in the figure below, where is 64 represented? What part does represent 160?

Break the process into all the steps.

16
x 14
24
40
60
100
224



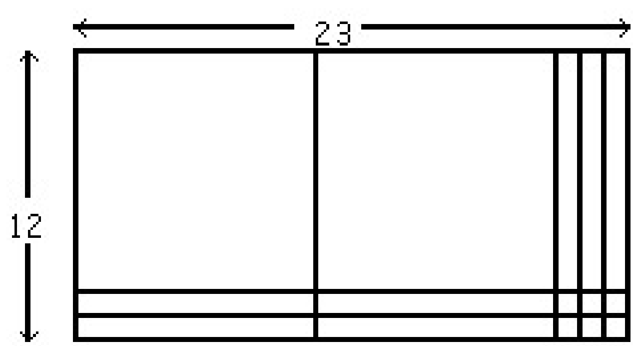
How is this process related to the use of the distributive property of multiplication over addition?

(10 + 6) x (10 + 4)

Another example, 12 × 23

Explain what partial products do each of the following numbers represent 6, 40, 30, 200. Identify the corresponding areas in the rectangle below.

23	20 + 3
× 12	× 10 + 2
46	40 + 6
230	200 + 30
276	200 + 70 + 6



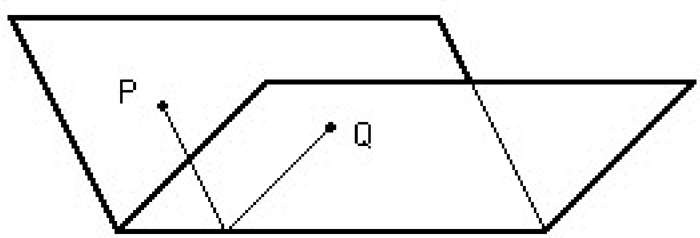
Paper Folding Geometry

Activity 1. A straight line

Wax paper works best to see the creases. Patty paper also works great and is very practical.

Basic constructions
A straight line

- Mark two points P and Q on the patty paper.
- Fold over a portion of a sheet of paper so that point P is placed on top of point Q.
- Hold two points P and Q tightly together using a finger and the table, and crease the paper with the other hand.
- Extend the crease in both directions to form a straight line.
- Mark one new arbitrary point R on the crease.
  What can you say about the distance of R to P and to Q?
- Fold along the crease again to verify.
- Use other points along the crease to convince yourself that the points on the crease are equidistant from P and Q.

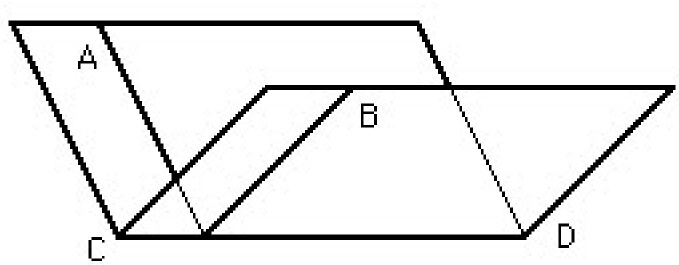


- Draw the segment connecting the points P and Q.
  What is the angle formed by the drawn segment and the crease?
  Where does the crease intersect the segment?

Activity 2. A straight line perpendicular to a given straight line

A line that is perpendicular to a segment and divides the segment in two equal parts is called the perpendicular bisector of the segment.

- Draw a segment and make a crease that divides the segment in half and is perpendicular to it.
- Draw or fold a line AB on patty paper.
- Fold the sheet over so that a segment of the given line AB is folded over onto itself.
- Holding the lines together with the fingers of one hand and the table, form a crease with the other hand.

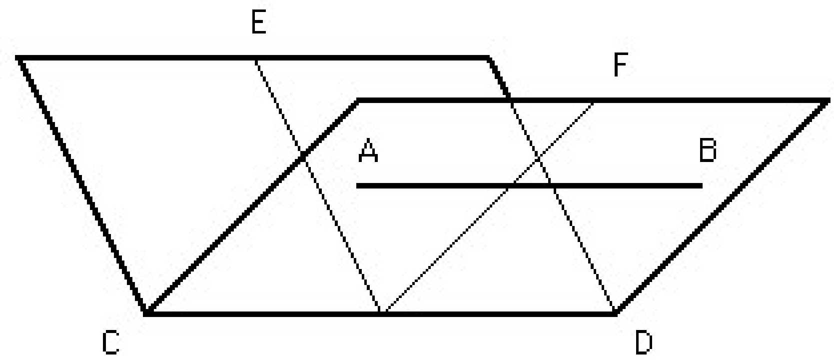


- Observe the angle between the original line and the new crease.

Paper Folding Geometry

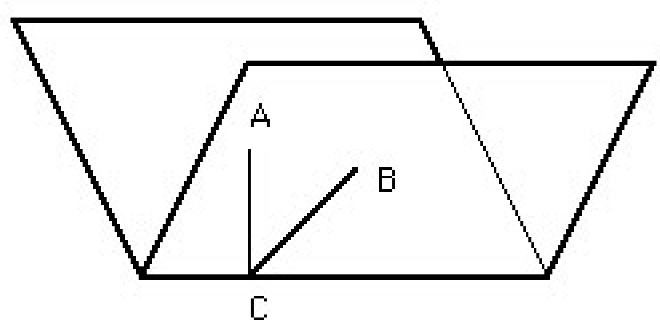
Activity 3. A parallel line to a given straight line

- Construct EF perpendicular to AB, and then construct CD perpendicular to EF.
  What can you say about AB and CD?



Activity 4. Bisect an angle

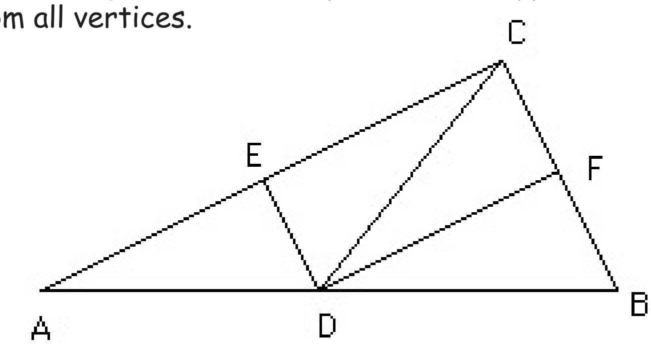
- Trace an angle ACB (that is the vertex of the angle is at C).
- Fold and crease the paper so that the legs CA and CB of the given angle coincide on top of each other.
- Open the paper and look at the two smaller angles formed.
  What can you say about them?



Other activities folding paper.

Activity 5. The midpoint of the hypotenuse

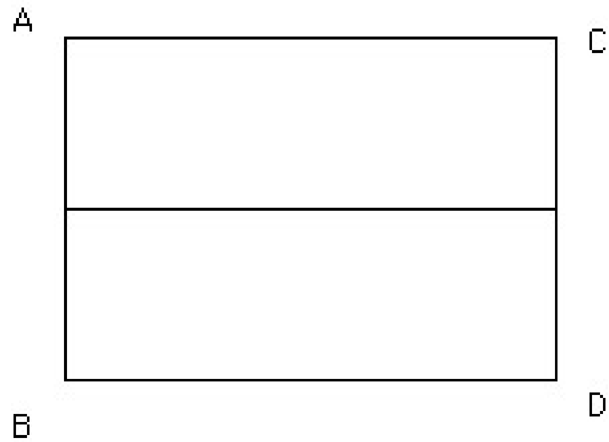
- Cut a right triangle (a triangle with a 90° angle).
- Find the midpoint of the largest side (this side is called the hypotenuse).
- Show by folding the triangle that the midpoint of the hypotenuse of a right triangle is at the same distance from all vertices.



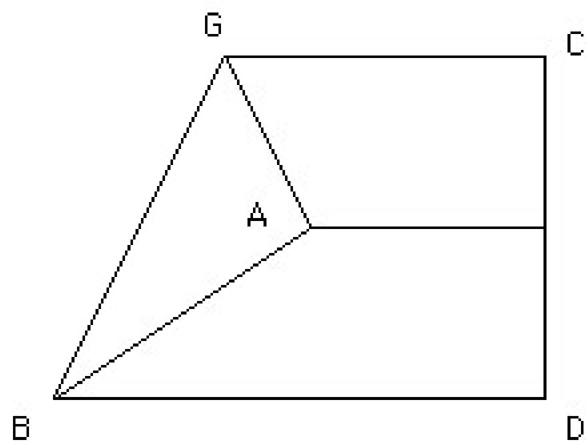
Paper Folding Geometry

Activity 6. A 60º angle

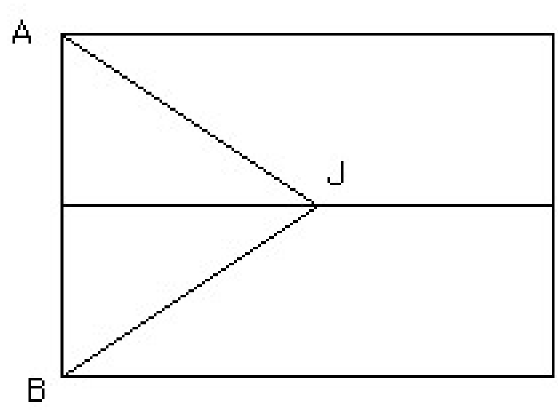
On a rectangular piece of paper ABCD fold the line that is parallel to two borders and is equidistant from them. You can do this by overlapping AC on top of BD. Open the piece of paper.



Fold vertex A onto the median so that the resulting crease GB passes through B.



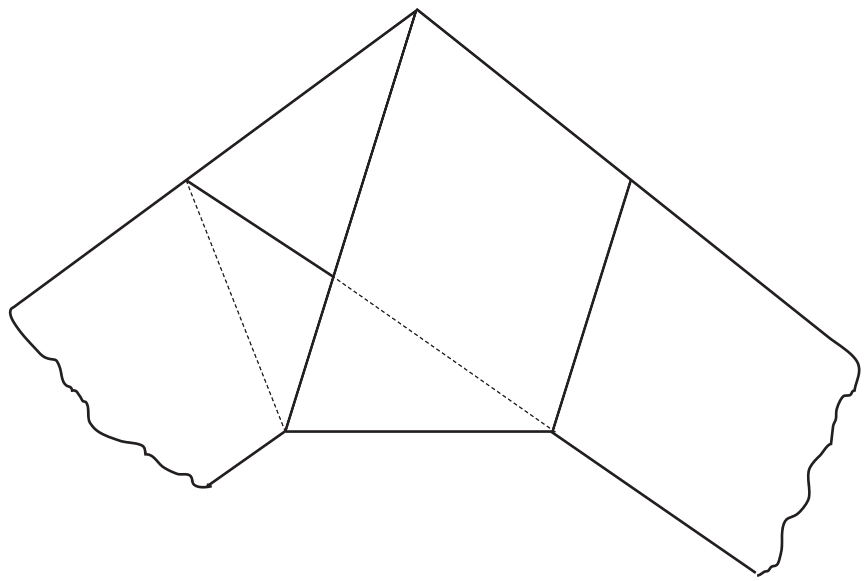
Denote by J the position of A on EF. Trace the segment JB. Provide convincing evidence or prove that angle ABJ is 60º



Paper Folding Geometry

Activity 7. A regular pentagon from a strip of paper

Use a long strip of constant width (adding machine paper works very well). Tie a knot. Tighten the knot and crease it flat.



Reference

Johnson, Donovan. *Paper folding for the mathematics class*. Reston, VA: National Council of Teachers of Mathematics, 1995.

Outdoor Activities for Circumference

Activity 1. A circle with people

Form a circle with 3/4 of the participants (for example, if there are 16 people, 12 will form the circle). They hold hands and stretch out. Then have 1/4 of the participants form a diameter inside the circle (with 12 people forming the circle, 4 would go inside).

- *What is the ratio of the number of people on the circumference to the number of people on the diameter?*

Activity 2. How far is it around a circle?

(Adapted from Lovitt & Clark, p. 209)

Materials needed: One eight-meter length of string for each group of three people.

1. Demonstrate to the whole class

Three people are needed with a string. The person who stays at the center stands still and holds the end of the string. The walker stands beside the center person, and then takes four paces, letting out the string. The observer marks the starting position, and then the walker paces and counts while the center person pivots so that he or she does not get entangled.

- *If the walker keeps walking forward keeping the string tight, what path will she make?*

Surprisingly, it is not always obvious to all people that the path will be a circle. It is an opportunity to include terminology such as radius and circumference.

- *How many paces do you think she will take around that circle?*

The guesses are an indication of students' perceptions. The correct answer is approximately 25. The range of guesses is amazing and worth pointing out to the group. Note how many guesses are fewer than 25, and how many are over. If participants make a guess, they will usually be interested in the accuracy of that guess.

2. Group work

It is more beneficial for all participants to experience the walk themselves, especially those whose guesses indicated a rather poor perception. Participants break into groups of three and work through the problems, including the first four-pace problem, and some they make for themselves. Point out that they should make a guess before each walk, and take turns at being center and walker.

3. Discussion

When most of the groups have completed the table, group the class together.

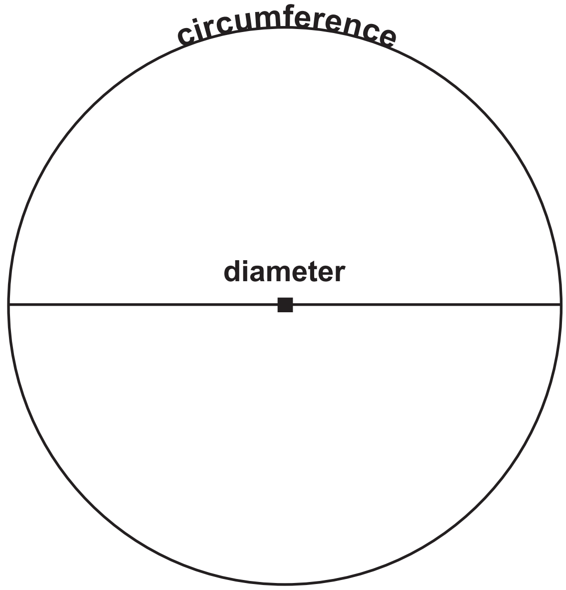
- *Were your guesses becoming more accurate?*
- *Can you see a relationship between the distance around the circle and the distance out from the center?*
- *Does this relationship work in all cases?*

It seemed to be about six, even with big steps. During the trials, participants were interested in the effect of using consistently small or large steps. They were intrigued to discover that the relationship did not change as step size changed.

There may be much discussion about the different answers the participants get, and about the errors of measurement that could produce these. Most results were $C = 6r$, or $C = 7r$. However, the accuracy of the value is not important to this lesson. What is important is the existence of the proportional relationship between C and r. The overall result is that each person would recognize that the circumference is approximately six times its radius. This is a good basis to appreciate the formula $C = 2r\pi$.

Relation Between Circumference and Diameter

The diameter of a circle is two times the radius. It is also the longest distance across a circle.



Materials. For these activities you will need cans, flasks, circular lids, or other objects that have a circle, adding machine paper, cm measuring tape, and a calculator.

Activity 1

Cut a strip of adding machine paper that measures the distance around the can. Use another strip of paper to measure the distance across. Display the two strips next to each other. Compare the lengths of the two strips.

- *What can you say about their relative size?*

Repeat the activity with other objects that have circles of different sizes. For each circle, compare the strips of paper corresponding to the diameter and to the circumference.

Activity 2

Measure using a cm measuring tape the circumference of the can. Measure the diameter. Enter your value on the table.

Repeat the activity with other objects that have circles of different sizes.
Observe the values you obtained on the different objects.

- *What can you say about the size of the circumference compared to that of the corresponding diameter?*

Use a calculator to compute the ratio of the circumference to the diameter. Report the value to one place after the decimal point.

Table for Diameter, Circumference, and Ratio

Diameter d	**Circumference c**	**Ratio c / d**

Here is a table with values obtained by a group of participants. Compare it to your own table

Diameter d	**Circumference c**	**Ratio c / d**
16.5	50	3.0
9	28	3.1
7.5	24.5	3.2
7	21	3.0
5.5	18	3.2
10	31.4	3.1
4.5	13.5	3.0
9.4	28.7	3.0

Notice that in each case the ratio c / d is a little over 3. More precisely, it is about 3.1. This ratio does not depend on the size of the circle. The value of the ratio c / d is called π. Because of measurement error, we are not able to compute exactly the value of π by measuring. The exact value of π can be computed using other methods. For most practical purposes, approximating π as 3.14 is quite appropriate.

The Area of a Circle

Activity 1. Counting Unit Squares

As we saw before, the area of a flat figure is given by the number of unit squares that can fit inside. So in principle, you could determine the area of a circle by using a grid. You would have to count how many squares fit completely inside, and how many additional fractions of squares you also need to count (see figure 1).

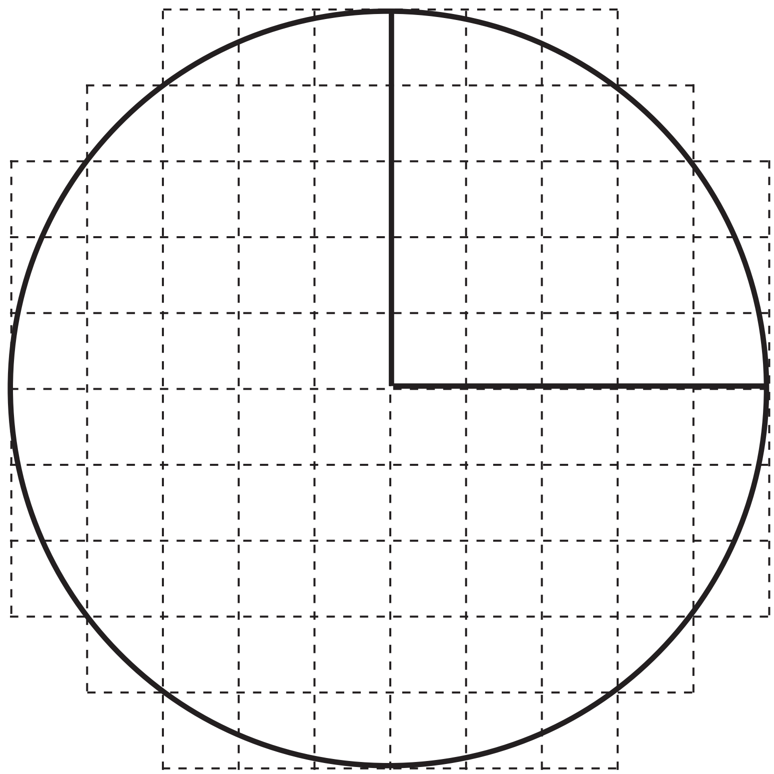


Figure 1. Circle in grid of square cm.

This approach, although conceptually very illuminating, has the disadvantage that when we actually count squares the estimation obtained thus is not very exact.

Exercise

Count the number of unit squares contained in a quarter of the circle. Make sure you include in your count fractions of squares contained. Multiply by four to estimate the total area of the circle.

In principle we could obtain better estimations by using a finer grid. The error would be less, but the disadvantage is that it would be too time consuming. Instead, we will use a different strategy to compute the area of a circle, by comparing the total area to the area of a radius squared.

Circle in Grid

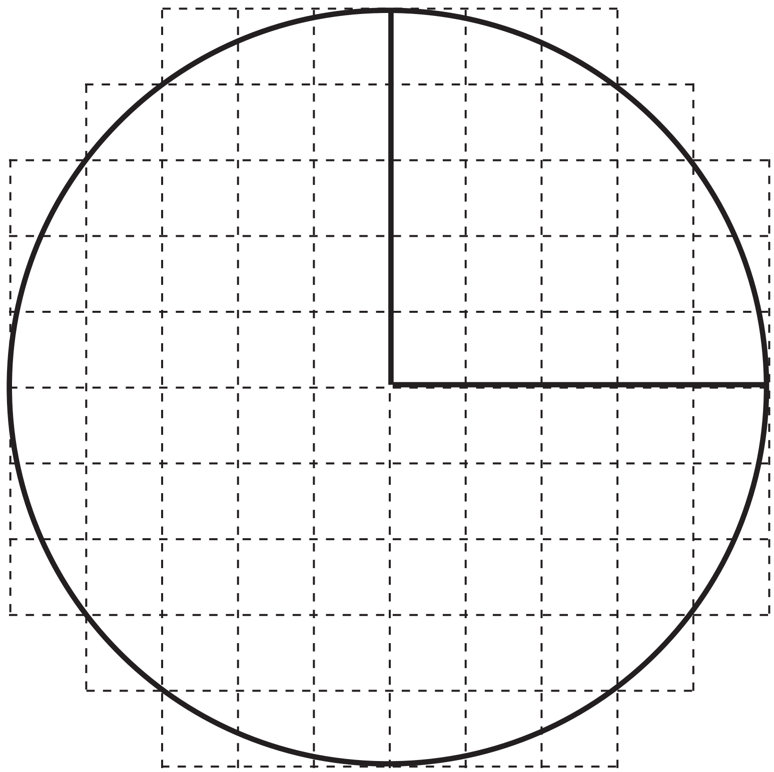


Figure 1. Circle in grid of square cm.

Activity 2. Radius Square

First estimate of the ratio of the area of the circle and the radius square. Guess how many times does the radius square fit into the circle.

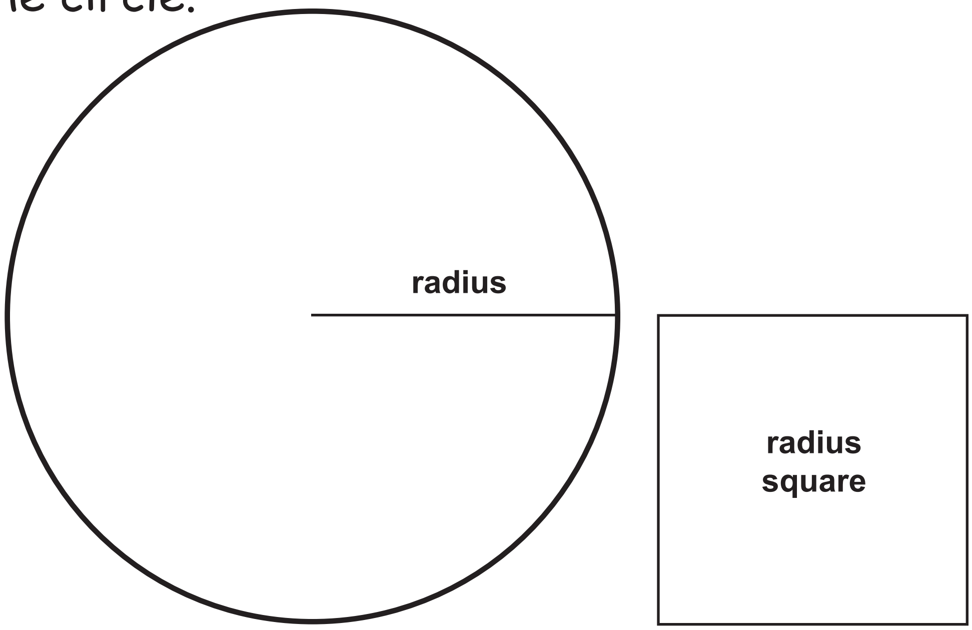


Figure 1. The circle and the radius square

Use figure 2 to convince yourself that the radius square fits more that two times in the circle.

Does the radius square fit four times into the circle?

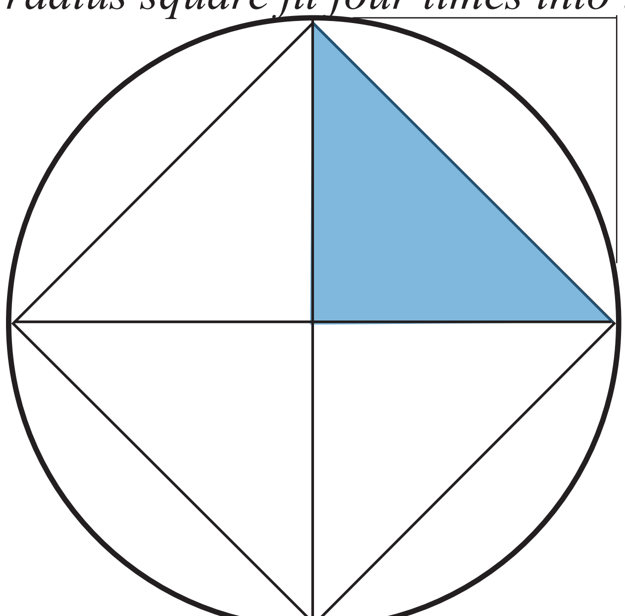


Figure 2. The shaded triangle is one half of the radius square

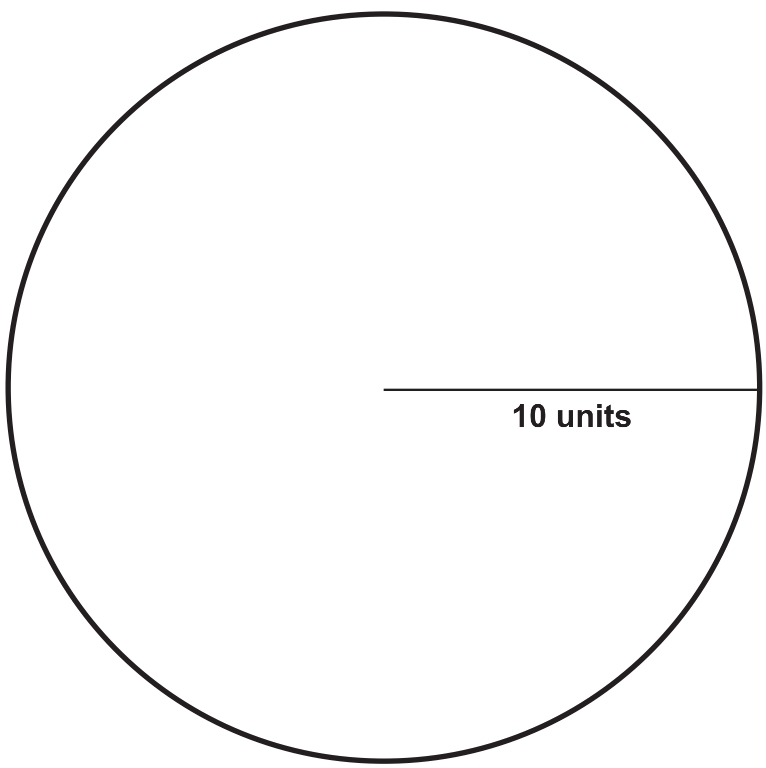
Circle with Radius

Activity 2. Radius Square

1) Cut out the circle.
2) Cut out the first colored square and fit it entirely inside the circle.
3) Cut out the second colored square and fit it inside without overlapping the previous color. You will have to cut parts of the second square so that there are no parts hanging out, and fit them inside the circle. You can fit the pieces of the squares in other ways. Use the entire second square before using the third colored square.
4) Continue with the third colored square. Make sure there are no overlaps and that there are no parts hanging outside the circle.
5) After you are done fitting the third colored square inside the circle, see whether there is still room for part of the fourth square (the one with the grid).
6) Save the remainder of the fourth square and use the grid to estimate how much of the fourth square you were able to fit.
   - *How many squares were you able to completely fit inside the circle?*
   - *How much of the last circle were you able to fit?*
   - *Count how many of the little unit squares of the fourth radius square were actually used.*
   - *Describe the relation between the area of the radius square and the area of the circle in your own words.*
   - *Based on this activity, what would be an estimation of the value of the ratio of the areas of the circle and the radius square?*

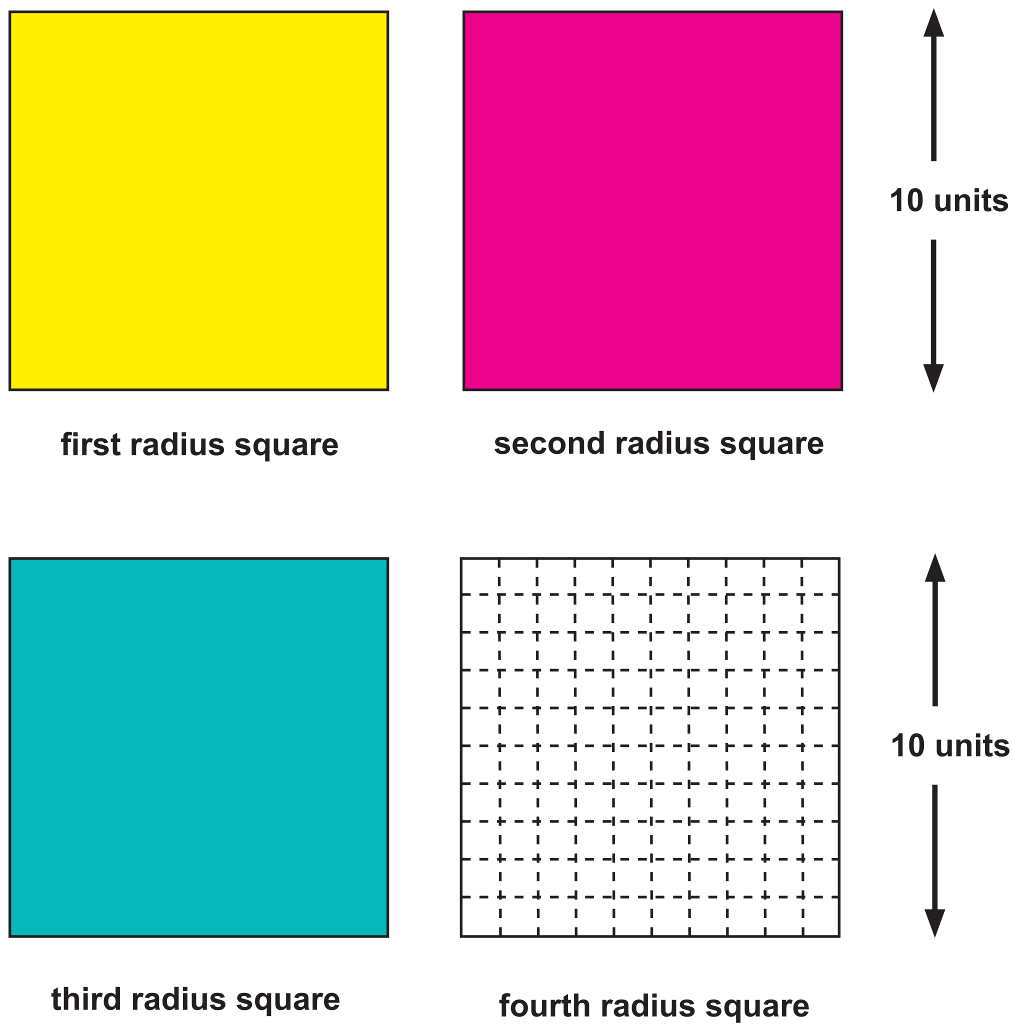
Materials

The squares provided are radius squares of the given circle. Cut the squares and cut out the circle. Follow the instructions for Activity 1.



Square of the Radius

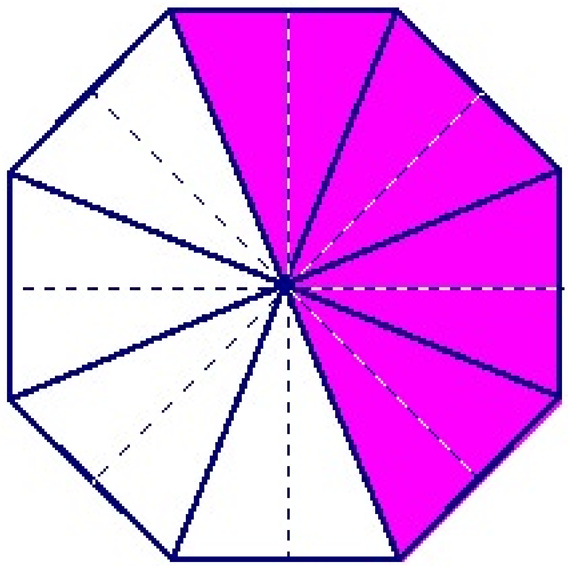
Activity 2. Radius Square (continued)



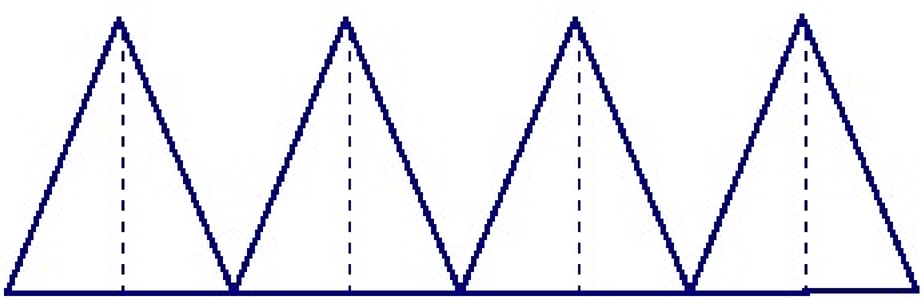
Area of Regular Polygons

The vertices of this regular polygon have been connected to the center (continuous lines), forming triangles. The dotted lines mark the perpendicular distance from each side to the center. This distance is called the apothem. Notice that half of the triangles are one color and the other half of the triangles are a different color.

- *How many sides does the polygon have?*
- *How many triangles are formed?*
- *How many of each color?*

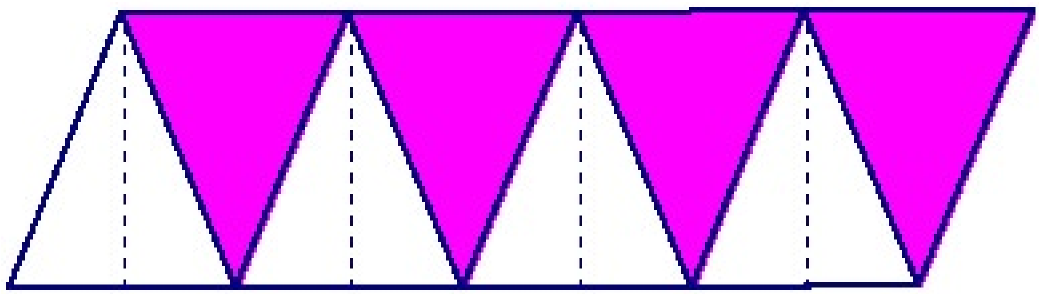


a) Cut the polygon into triangles along the continuous lines (NOT the dotted lines) and arrange the triangles of one color so that their bases are on the same line.



b) Put the triangles of the other color "upside down" to fill in. You will form a parallelogram with the triangles of two colors.

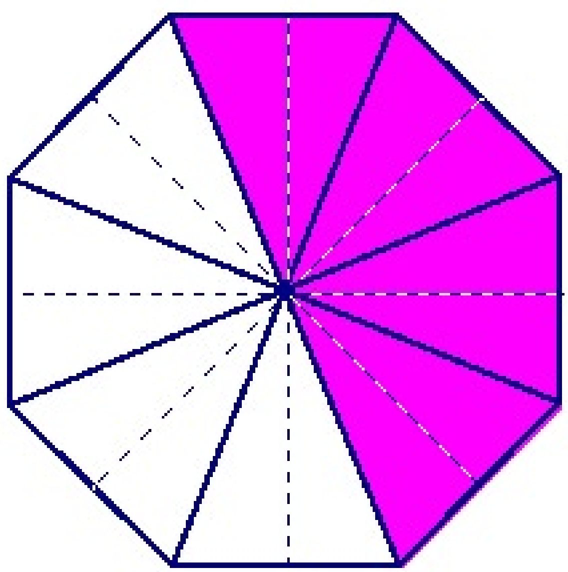
Area of Regular Polygons



c) Compare the length of the base of the parallelogram with the perimeter of the original polygon.
d) Write an expression for the base of the parallelogram in terms of the perimeter of the polygon.
e) Compare the height of the parallelogram with the apothem of the original polygon.
f) *What is the area of the parallelogram compared with the area of the polygon?*
g) Write an expression for the area of the parallelogram in terms of the perimeter and the apothem.
h) Write an expression for the area of the original polygon in terms of the perimeter and the apothem.

Materials

Cut the triangles along the solid lines.



Connecting Formulas for Area and Circumference

Figure 1 shows a circle of radius r. Its circumference can be computed by using the formula 2 x r x π. The circle is divided into 12 equal sectors. Half of them have been colored, the other half are white.

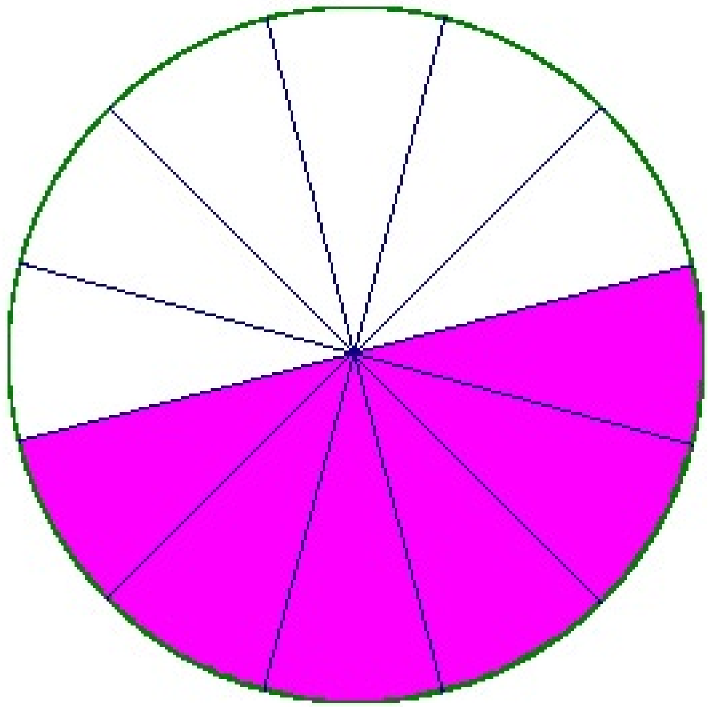


Figure 1. A circle cut into 12 slices.

Imagine that you cut out the 12 sectors and rearrange them to form a shape that resembles a parallelogram as shown in figure 2. The area of the circle is the same as the area of this shape. To compute the area of this shape we will use the fact that it resembles a parallelogram, although its "base" is not quite a straight line. We will use the formula for the area of a parallelogram, Area = base x height.

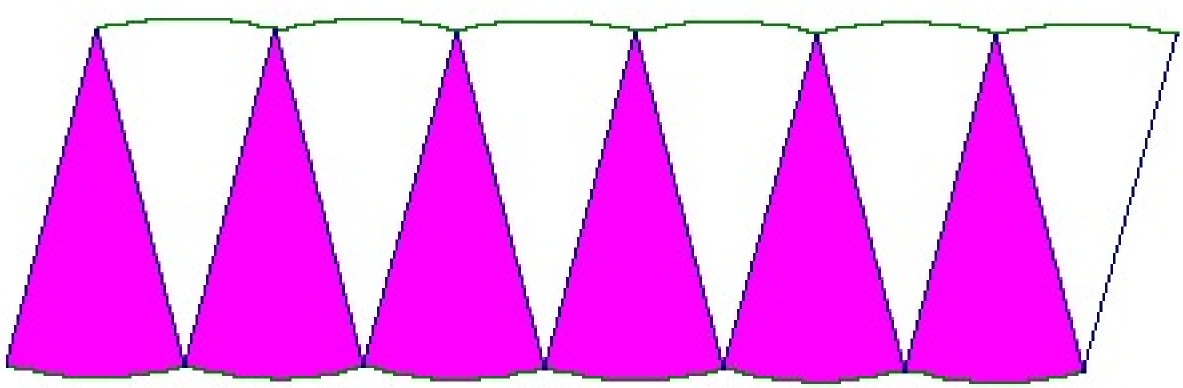


Figure 2. The slices of the circle rearranged.

Connecting Formulas for Area and Circumference

1) *What is the sum of the lengths of the little arcs that form the "base" of this "parallelogram" compared with the circumference of the original circle? Why?*

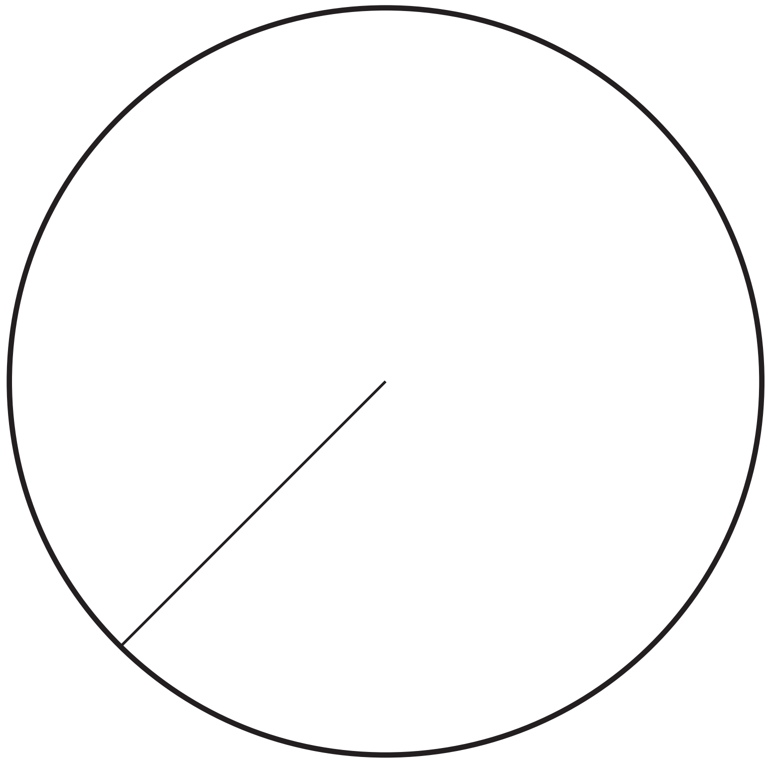
2) Write an expression for half the circumference in terms of the radius of the circle.

3) Explain why the sum of the arcs of the "base" of the rearranged shape is r π.

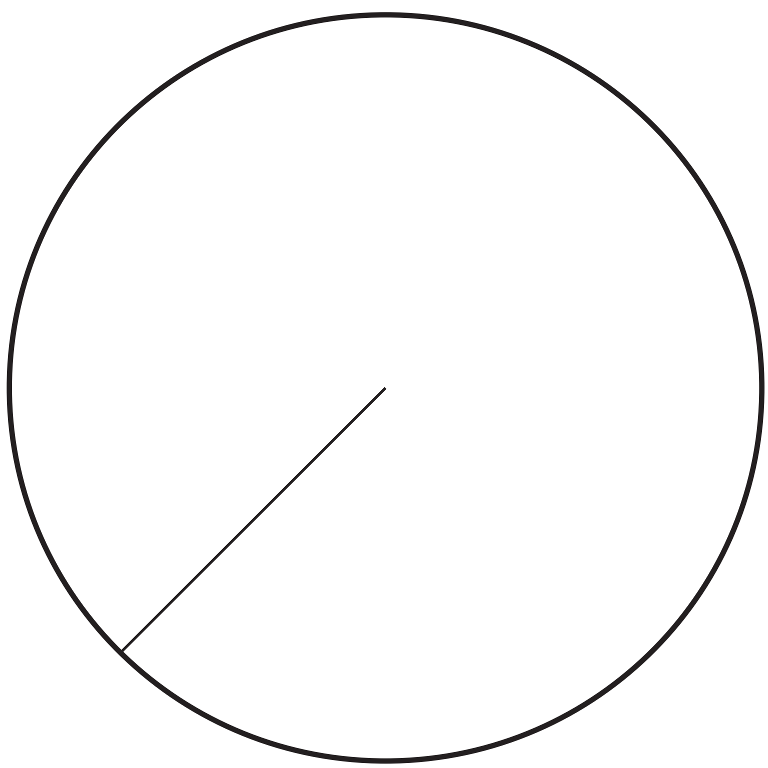
4) Compare the height of the "parallelogram" with the length of the radius of the original circle.

A Cone from a Circle

Cut out the circle and along the marked radius.

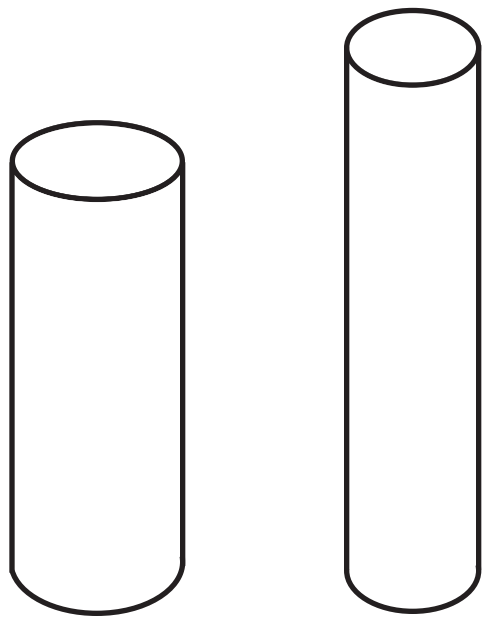


Cut out the circle and along the marked radius.



Cylinders with the Same Lateral Area

Take two identical sheets of paper, 11 inches by 8.5 inches. Form a tall cylinder with one of them by joining and pasting the two long edges of the rectangle. With the other sheet form a shorter cylinder by joining and pasting the shorter sides of the rectangle. The cylinders have only the lateral surface, with no base and no top. That is they look like a pipe rather than an unopened can.



Two cylinders formed with the same rectangles.

a) *Which of the two cylinders do you think will have the bigger volume?*
b) *Make an experiment and fill the long skinny cylinder first with packing material and then pouring the content into the other cylinder. Many people are surprised by the result. What have you observed?*

In what follows we will see how we can compute the volumes of the two cylinders to understand the result.

c) *What is the circumference of the base of the tall cylinder?*
d) *What is the height of the tall cylinder?*
e) *What is the circumference of the base of the short cylinder?*
f) *What is the height of the short cylinder?*
g) *Compute the radius of the base of each cylinder.*
h) *What is the area of the base of each cylinder? Remember that the area of a circle is given by* πr^2*, which is multiplying the radius by itself and then by 3.14.*

The volume of a cylinder can be computed by multiplying the area of the base times the height.

i) *What is the volume of each of the cylinders?*
j) *Are your results consistent with the result of the experiment?*

"Computing the Circumference" and "The Reverse Problem"

Supplementary material for groups that move at a faster pace than other groups in the class.

Computing the Circumference

If the diameter of a circle is known, we don't need to measure the circumference, we can compute it. We saw that for any circle the value of the ratio c / d = 3.14. We can use this information to compute the circumference if the diameter is known, $c = d \times 3.14$ or $c = d \times \pi$.

- Compute the circumference of a circle that has a diameter of 12 cm.
- Compute the circumference of a circle that has a diameter of 9 feet.

In some cases, it is the radius r of the circle that is known. We can use this information to compute the circumference. Remember that two times the radius is equal to the diameter, $2 \times r = d$. Therefore, the circumference that is equal to $d \times 3.14$ or $d \times \pi$ will be equal to $2 \times r \times 3.14$ or $2 \times r \times \pi$.

- Compute the circumference of a circle of radius 2.5 cm.
- *What is the circumference of a circle of radius 6 ft?*

The Reverse Problem

Computing the diameter given the circumference.

In some instances it is easy to measure the circumference, but harder to measure the diameter or the radius, for example, in the case of a tree. In this case we can compute the diameter by dividing the circumference by π. For example, if the circumference is 23 cm, the diameter will be $23 \div \pi$ or quite approximately $23 \div 3.14 = 7.3$

- Compute the diameter of a circle that has a circumference of 4 ft.
- *What is the radius of a circle that has a circumference of 6 ft?*

A "Thought" Experiment

Approximating the circle by regular polygons.

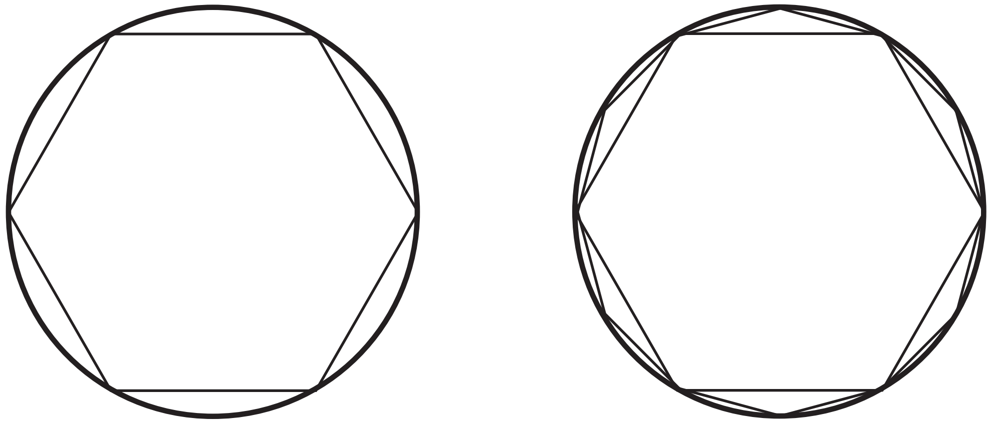
Imagine you have a family of regular polygons inscribed in the same circle constructed in the following way. Starting with a regular hexagon (figure 1a), the next polygon will have 12 sides. Six of the vertices will be common with the hexagon; the additional vertices will be the midpoints of the arches (see figure 1b). In the same way, each successive term of the family of polygons has twice as many sides. The perimeters of the regular polygons approximate better and better the circumference of the circle. Furthermore, by using a polygon with enough number of sides, we can make the difference between the perimeter of the polygon and the circumference as small as we want. The areas of the regular polygons approximate better and better the area of the circle. The difference between the area of the circle and the area of one of the polygons can also be made as small as we want by choosing a polygon with enough number of sides.

The area of the regular polygon can be computed by multiplying the perimeter times the height of one of the triangles forming the regular polygon (see figure 2), and dividing by two. This can be proved in several ways. One is to imagine all the triangles laid-out side by side (figure 3). The total area of the polygon is the sum of the areas of the triangles. One method to obtain the total is to compute the area of each triangle by multiplying the base times the height, divide by 2, and then add the areas. Or we can add all the bases first, which gives us the perimeter, then multiply by the height, and divide by 2. If the number of sides of the polygon is very large, the sum of the bases will be very close to the circumference of the circle ($2\pi r$), and the height of the triangle will be very close to the radius (r). Therefore the area of the polygon will be very close to $\frac{\text{circumference x radius}}{2}$.

Because the area of the circle and that of the polygons can be made as close to each other as we want, we have the area of the circle given by:

$$\frac{\text{circumference x radius}}{2} = \frac{\text{diameter x } \pi \text{ x radius}}{2} = \frac{\text{2 x radius x } \pi \text{ x radius}}{2} = \pi \text{ x radius}^2 .$$

A "Thought" Experiment



(a) The initial hexagon (b) New vertices at the midpoints of the arches

Figure 1. Regular polygons that approximate a circle.

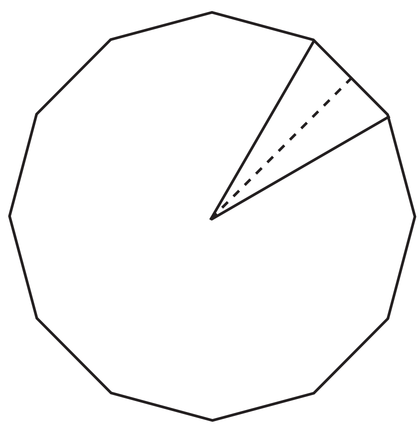


Figure 2. A triangle whose base is one side of the polygon.

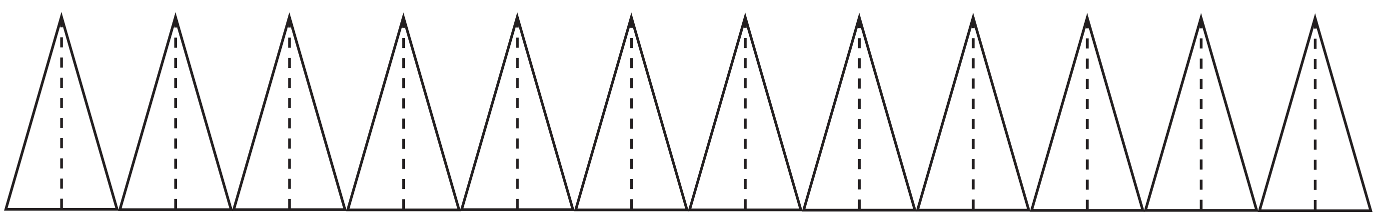


Figure 3. The polygon is broken into triangles.

We can also arrange the triangles that form the regular polygon into a parallelogram (see figure 4). Its base will be very close to half the circumference, $1/2 \times d \times \pi = 1/2 \times 2 \times r \times \pi$, that is $r \times \pi$, and its height will be very close the radius r of the circle. The area of the parallelogram will therefore be very close to πr^2. As the number of sides of the regular polygon increases, the height of the corresponding parallelogram gets closer and closer to the radius of the circle, and its base closer to πr. The area of the circle is given by $\pi r \times r = \pi r^2$.

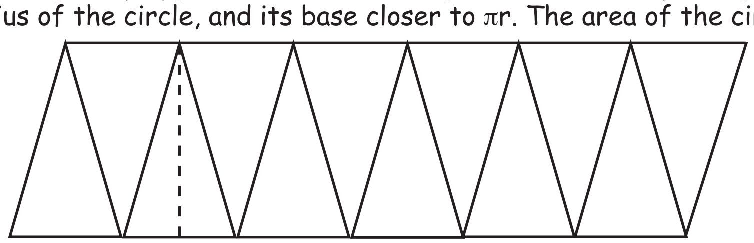
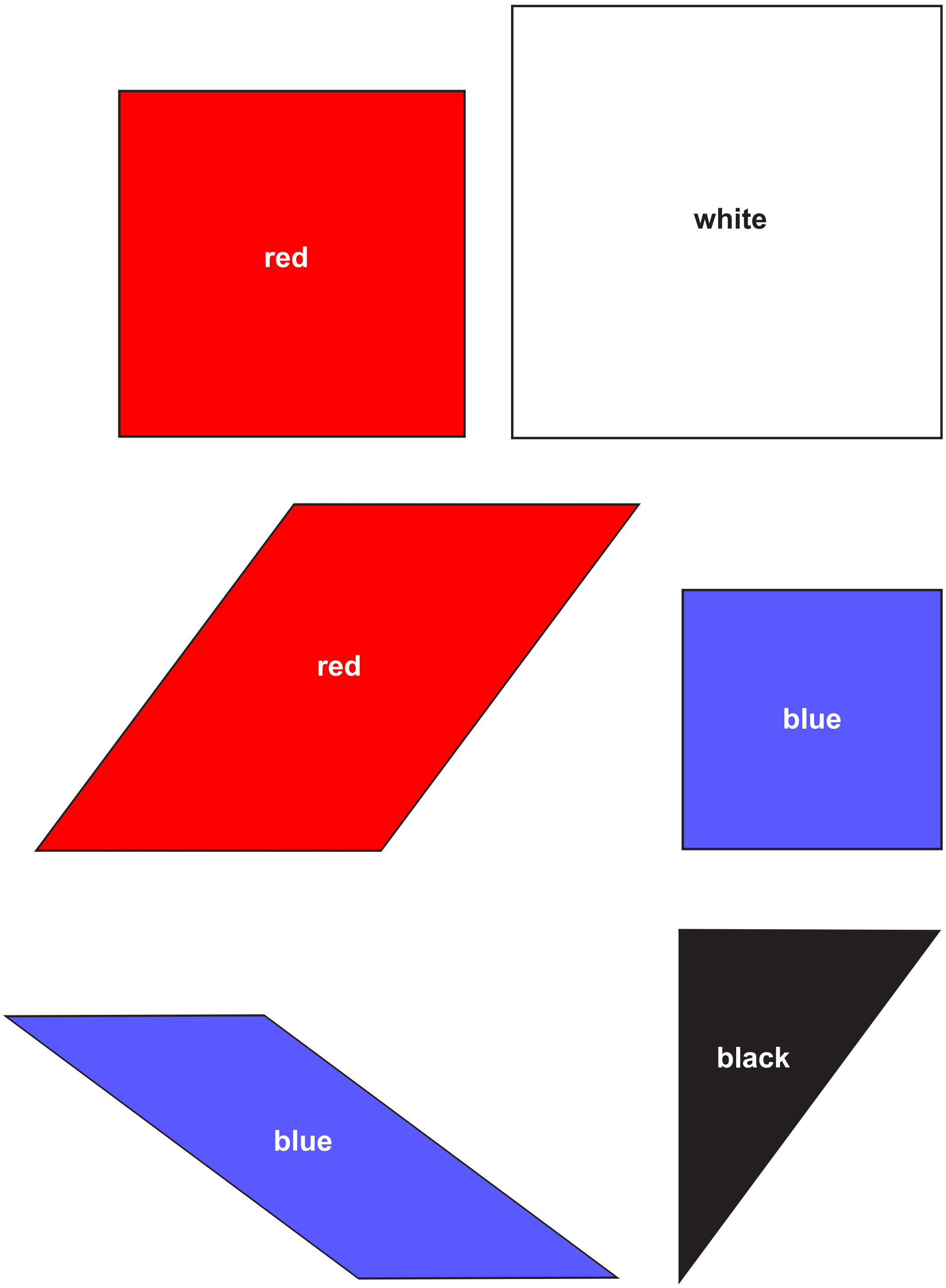


Figure 4. The polygon is rearranged into a parallelogram.

Pythagorean Puzzle Pieces: Set A

Paste on cardboard or copy on cardstock and cut out.



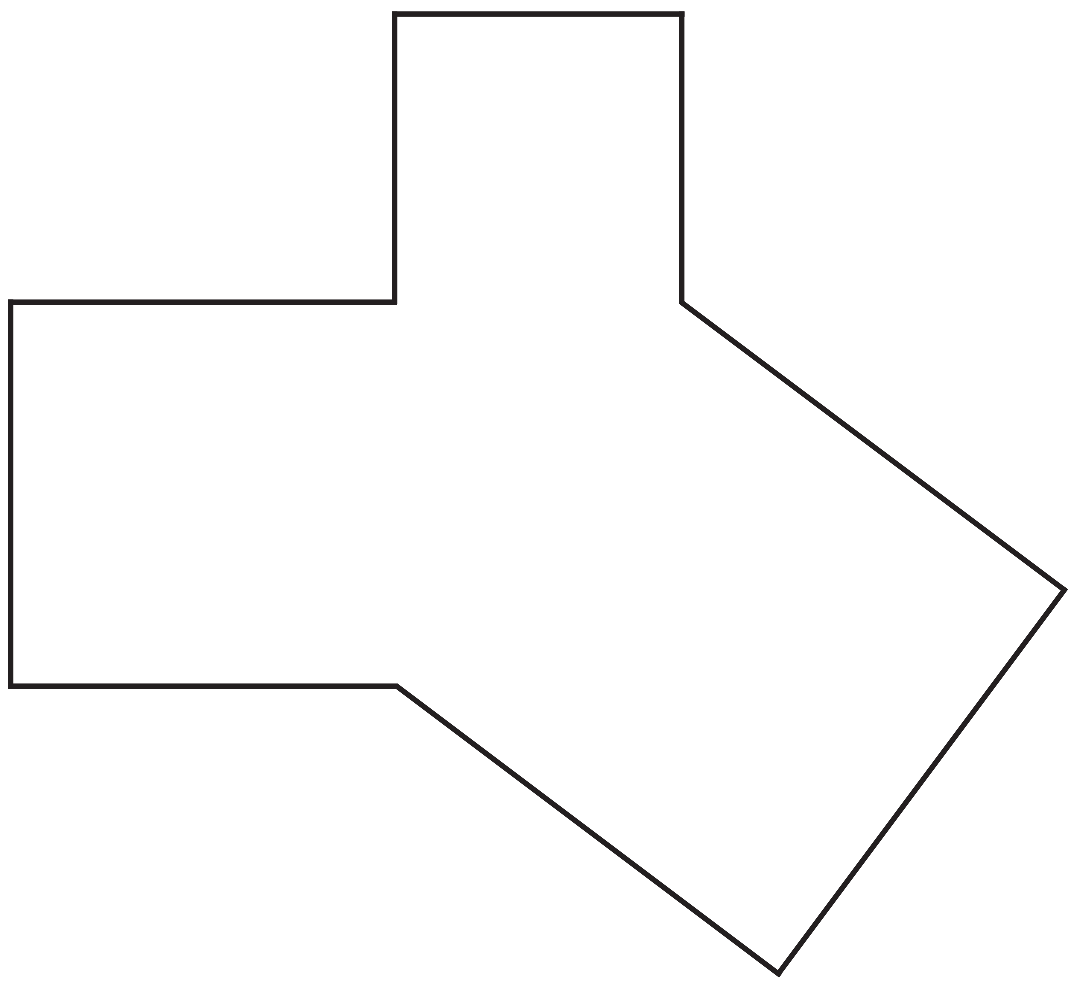
Pythagorean Puzzle A

Activity 1

Describe the cardboard pieces given.

What kind of triangle is used for this puzzle?

The sides that form the right angle are called the legs of the right triangle, the biggest side, the one that is opposite of the right angle, is called the hypotenuse.



Pythagorean Puzzle A - Activities

Activity 1

1) Use the triangle and the three squares to form the puzzle.
2) Compare the lengths of the sides of the squares with the lengths of the sides of the triangle.

Activity 2

1) Use the triangle, the small square, the biggest square, and the big parallelogram to form the puzzle.
   What can you say about the area of this parallelogram and the area of the square you did not use?
2) Verify that the base of the red square is equal to one base of the red parallelogram.
3) Compare the height of the red square with the height of the red parallelogram.
   What can you say about the height of the square and parallelogram?

Activity 3

1) Use the triangle, the two bigger squares and the small parallelogram to form the puzzle.
   What can you say about the area of the small blue parallelogram and the area of the small blue square?
2) Take the two pieces that have the same area. Verify that the base of the blue square is equal to one base of the blue parallelogram.
3) Compare the height of the blue square with the height of the blue parallelogram.

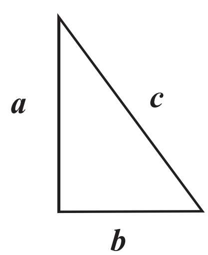
Activity 4

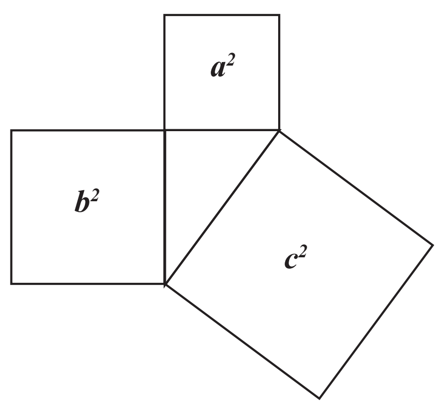
Use the triangle, the two parallelograms, and the two smaller squares to form the puzzle.
   What can you conclude about the area of the big square and the areas of the two parallelograms?

Activity 5

Link together the results obtained in activities 2, 3 and 4, to relate the area of the square on the hypotenuse to the sum of the areas of the squares on the legs. State the relation in your own words.

This is a very important theorem in mathematics, known as the Pythagorean theorem. If we label the hypotenuse as c, and the legs as a and b, the theorem can be expressed as $a^2 + b^2 = c^2$.



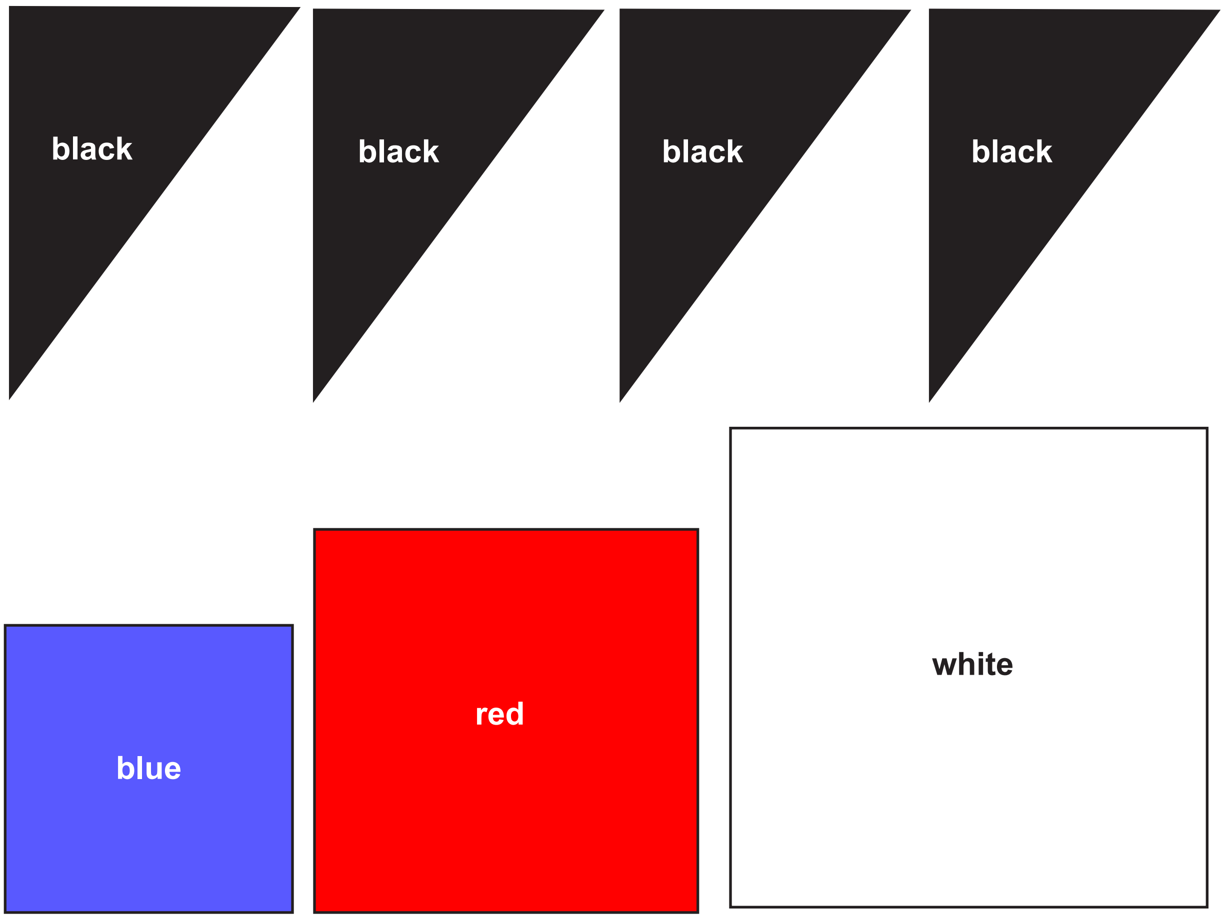


Reference

Hall, G. D. A Pythagorean puzzle. In *Teacher-made aids for elementary school mathematics: Readings from the Arithmetic Teacher*. National Council of Teachers of Mathematics, 1974.

Pythagorean Puzzle Pieces: Set B

Paste on cardboard or copy on cardstock and cut out.

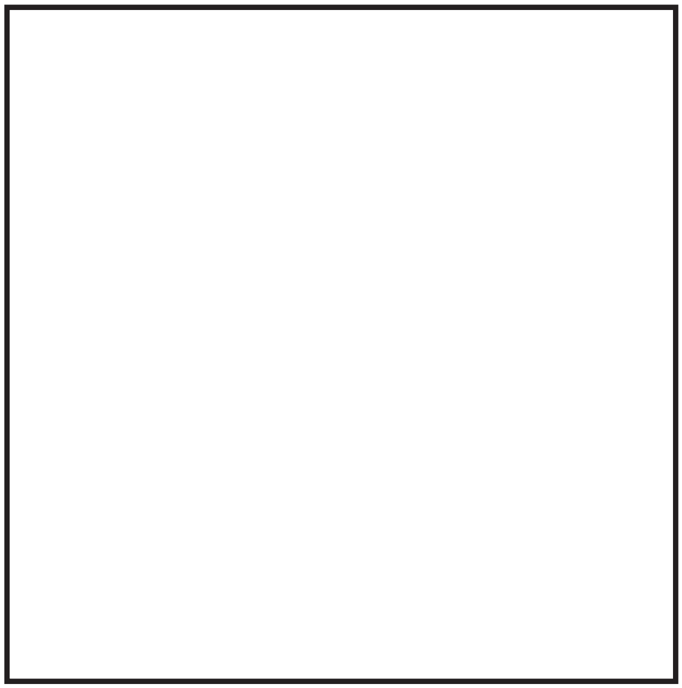


Pythagorean Puzzle B - Activities

Proof of the Pythagorean Theorem using four right triangles

Opening Activity

1) Compare the lengths of the given squares with the sides of the right triangle.
2) State your findings.



Activity 1

1) Use the four congruent right triangles, and the two smaller squares to fill the frame.
2) Express the area of the frame in terms of the areas of the pieces.

Activity 2

1) Now use the four triangles and the big square to fill the same frame.
2) Express the area of the frame in terms of the areas of the pieces.

Comparing Activity 1 and 2

What can you conclude about the area of the square on the hypotenuse and the squares on the legs?

Pythagoras on the Geoboard

Activity 1. Areas of Tilted Squares

The Pythagorean theorem can be used to compute the area of the squares shown in Figure 1.

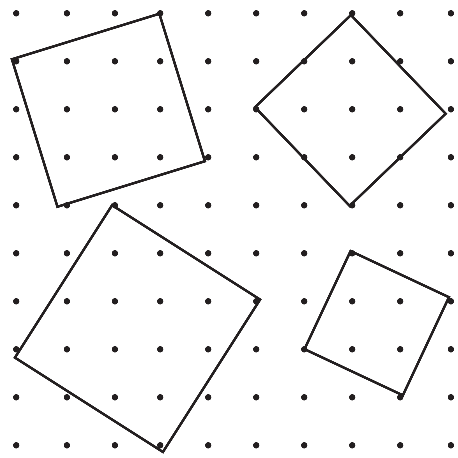


Figure 1

Given a tilted square on the geoboard, you can build a right triangle so that one side of the tilted square is the hypotenuse of the right triangle. Construct squares on the legs forming the right angle. Compute the areas of the squares on the legs and add them to obtain the area of the square on the hypotenuse. *What is the area of the square on the hypotenuse of the following figures?*

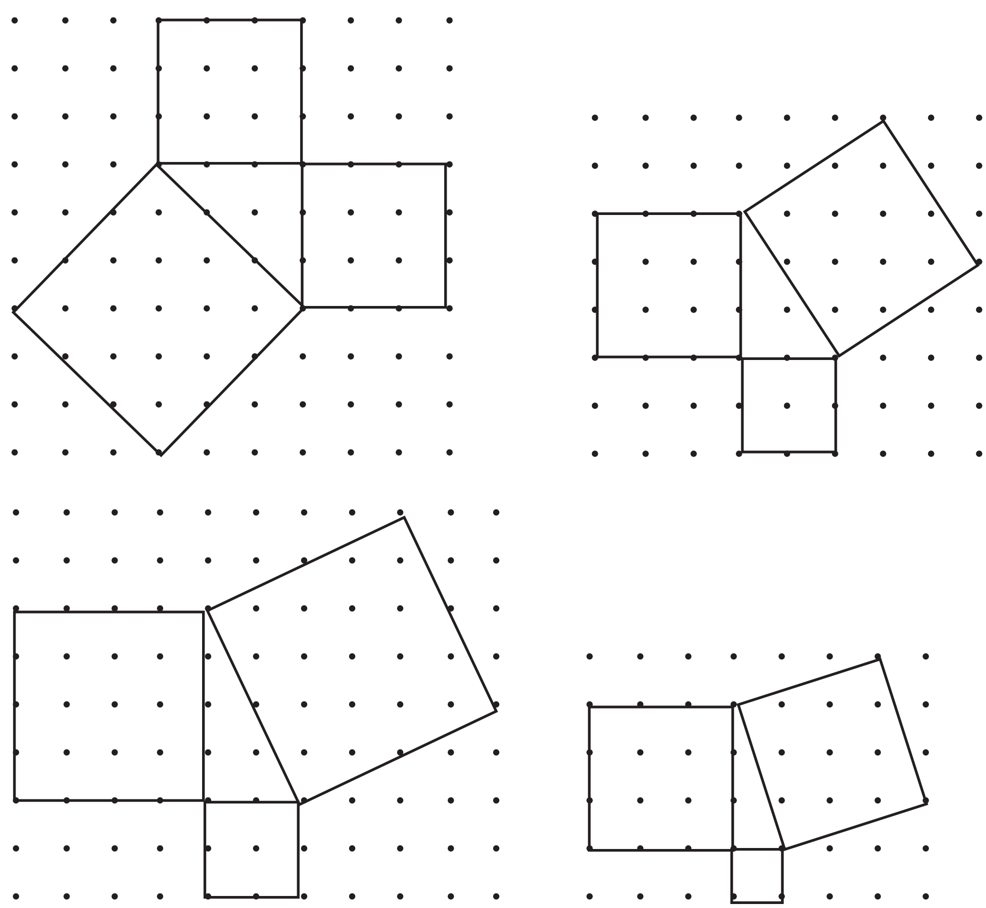
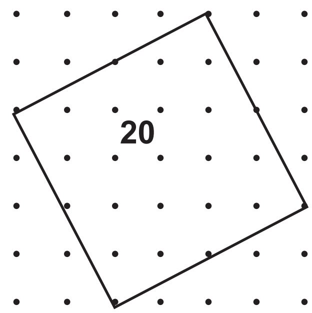


Figure 2

Pythagoras on the Geoboard

Activity 2. Determine the length of the sides of a tilted square.

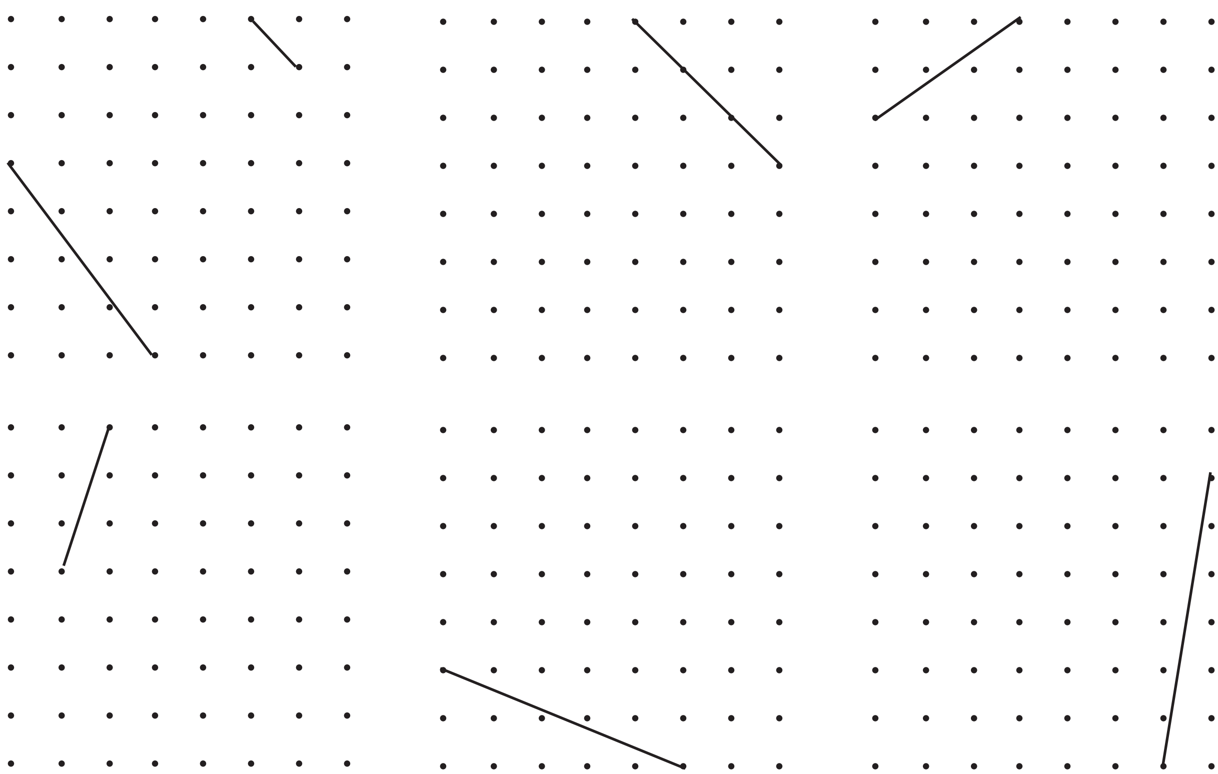
If we know the area A of a tilted square on the geoboard, we can find the length of its side. Remember that to obtain the area of a square is to multiply the length of the side by itself, that is $A = s \times s$ or $A = s^2$. For example, let us find the length of the side of a square of area 20. We need then to find a number that multiplied by itself is 20. We can see that this number has to be greater than 4, because $4 \times 4 = 16$. We also know it has to be smaller than 5 because $5 \times 5 = 25$. Calculators have a key to compute the square root of a number. Using the calculator we see that $\sqrt{20}$ is about 4.47.



Exercise. Use a calculator to determine the lengths of the sides of the tilted squares in the previous activity.

Activity 3. Distance between any two points on the geoboard.

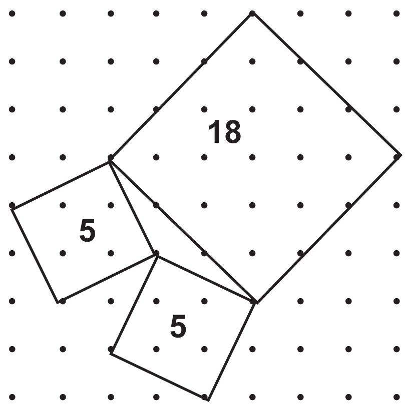
The distance between any two points on the geoboard can be computed by constructing a square that has that segment as its side. You can find the area of the square. The length of the side will be the square root of the area. Find the lengths of the segments shown.

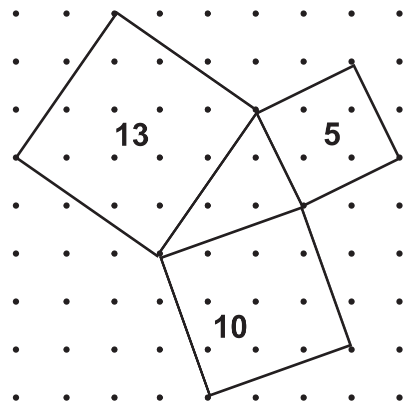


Inverse of Pythagorean Theorem

Supplementary material for groups that move at a faster pace than other groups in the class

We can use the geoboard to show that if the angle opposite to side c is greater that 90°, then $a^2 + b^2 < c^2$, and if the angle opposite to c is less than 90°, then $a^2 + b^2 > c^2$.





1) Construct a triangle on the geoboard that does not have a right angle.
2) Verify that $a^2 + b^2 \neq c^2$

Patty Paper Shapes

Activity 1

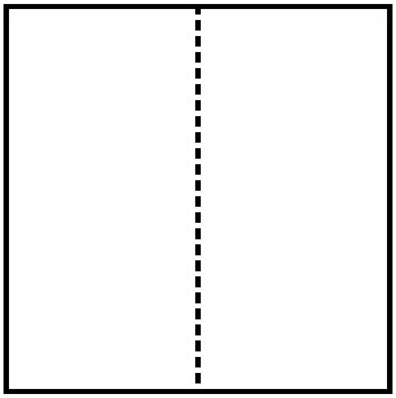


Figure 1a.
Square with
mid parallel fold

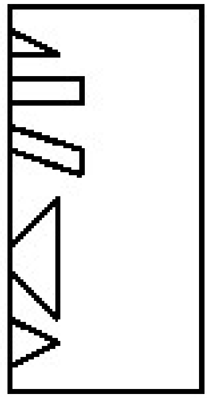


Figure 1b.
Cuts along
fold

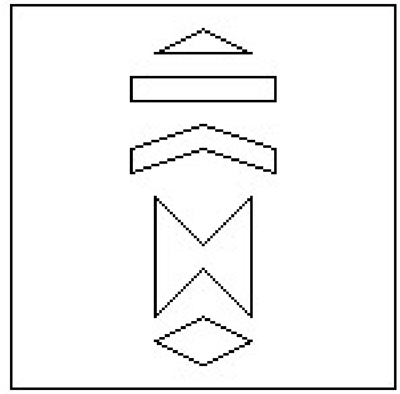


Figure 1c.
One axis of
symmetry

Activity 2

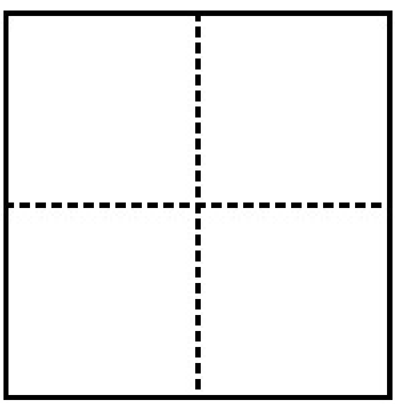


Figure 2a.
Square folded
in 4 parts

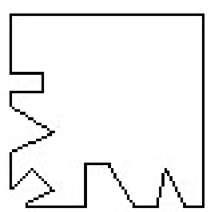


Figure 2b.
Cuts along
fold

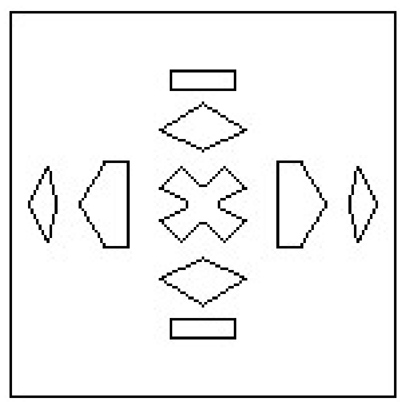


Figure 2c.
Two axes of
symmetry

Activity 3

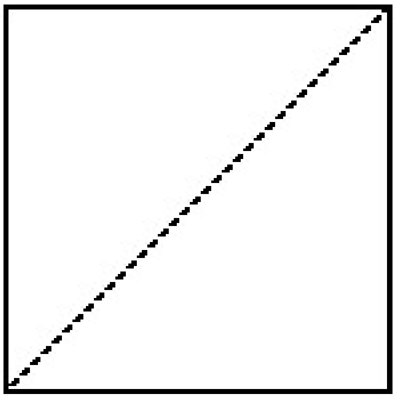


Figure 3a.
Square folded in half diagonally

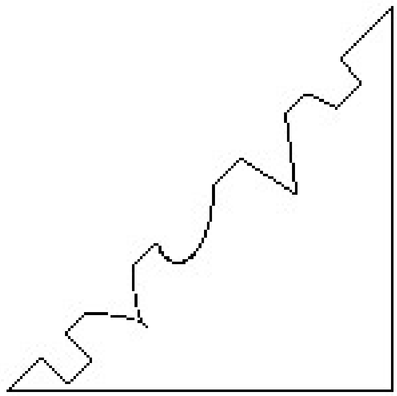


Figure 3b.
Cuts along fold

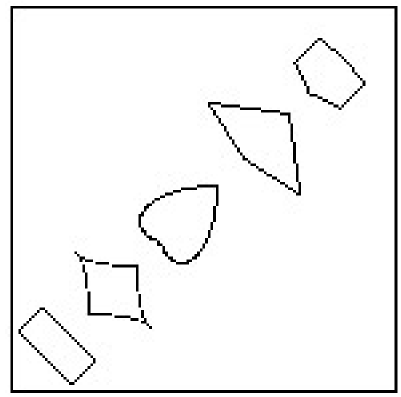


Figure 3c.
One diagonal axis of symmetry

Activity 4

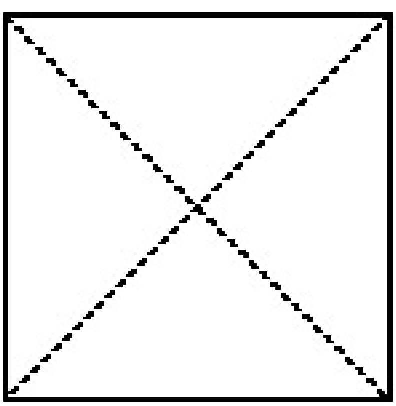


Figure 4a.
Square folded in 4 parts diagonally

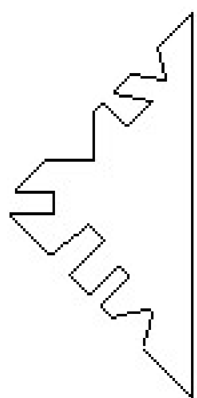


Figure 4b.
Cuts along fold

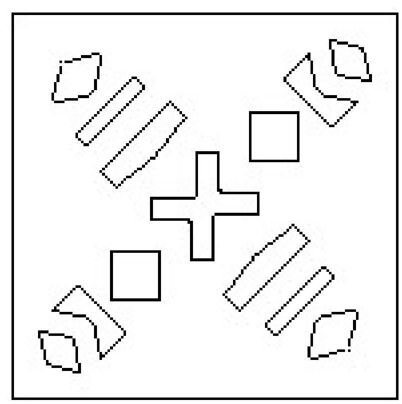


Figure 4c.
Two diagonal axes of symmetry

Activity 5

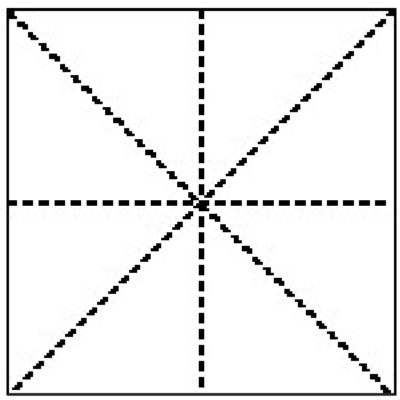


Figure 5a.
Square folded in 8 parts

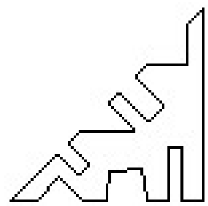


Figure 5b.
Cuts along fold

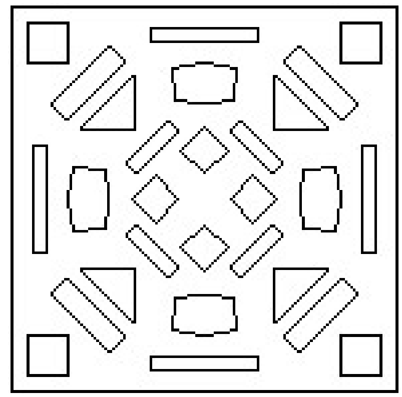


Figure 5c.
Four axes of symmetry

Patty Paper Activities

Materials. Scissors and squares of paper (Patty paper or tissue paper)

Activity 1. One axis of symmetry.

Fold square in half to form a mid parallel (figure 1a). With the square folded, use the scissors to cut triangles or other shapes along the fold and along the borders (figure 1b) . Predict what the paper will look like when you open it (figure 1c).

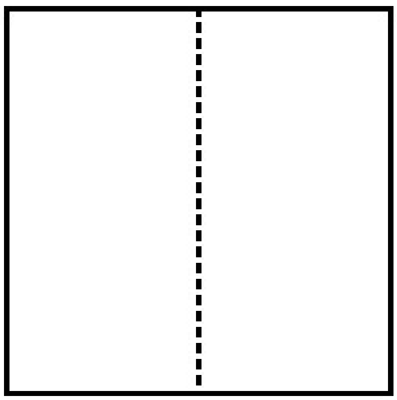


Fig. 1a. Square with mid parallel fold

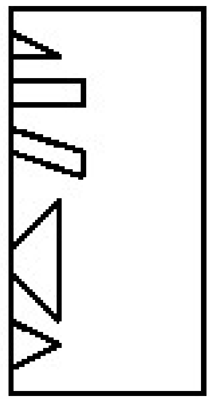


Fig. 1b. Cuts along fold

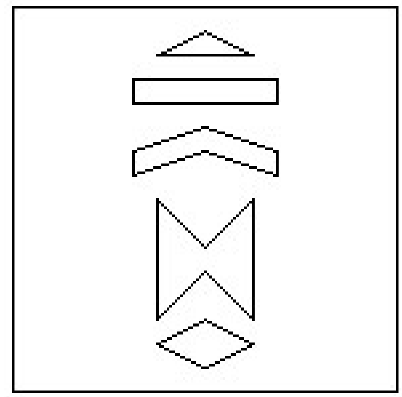


Fig. 1c. One axis of symmetry

Activity 2. Two axes of symmetry.

Fold a new square in half to form a mid parallel, and in half again so that the second fold is perpendicular to the first (figure 2a). With the square folded in four parts, use the scissors to cut triangles or other shapes along the folded creases or along the borders (figure 2b). Predict what the paper will look like when you open it (figure 2c).

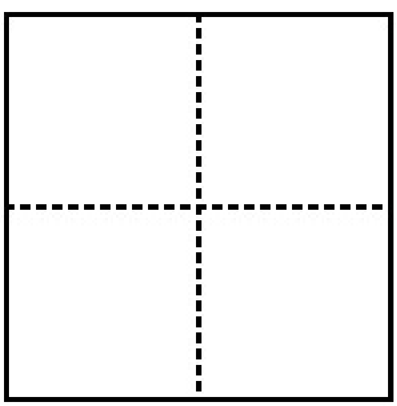


Fig. 2a. Square folded in 4 parts diagonally

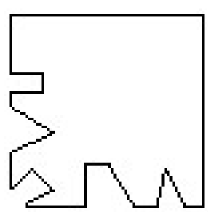


Fig. 2b. Cut along fold

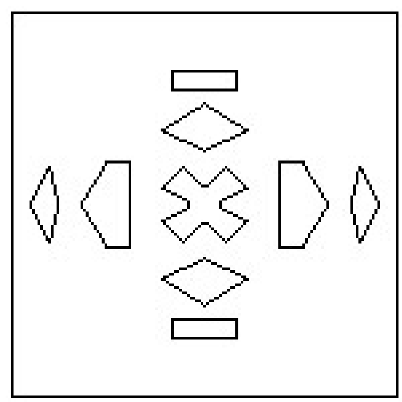


Fig. 2c. Two axes of symmetry

Patty Paper Activities

Activity 3. One diagonal axis of symmetry.

Fold a new square in half, this time along one diagonal (figure 3a). With the square folded, use scissors to cut little triangles or other shapes along the fold and the borders (figure 3b). Predict what the paper will look like when you open it (figure 3c).

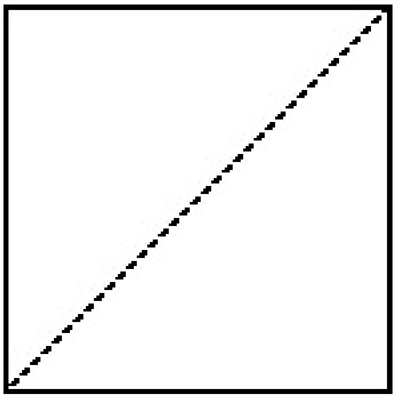


Fig. 3a. Square folded in half diagonally

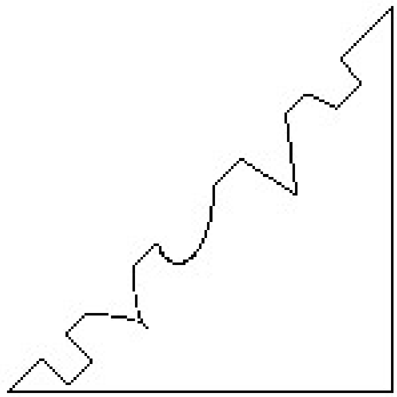


Fig. 3b. Cut along fold

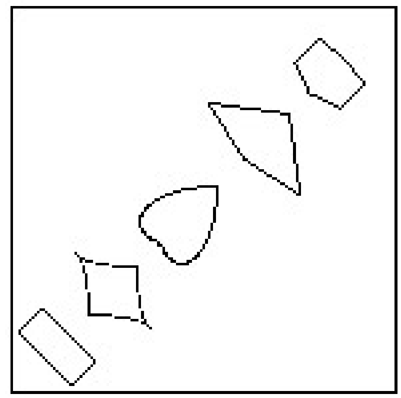


Fig. 3c. One diagonal axis of symmetry

Activity 4. Two diagonal axes of symmetry.

Fold a new square in four along the two diagonals (figure 4a). With the square folded, use scissors to cut triangles or other shapes along the folds or along the borders (figure 4b). Predict what the paper will look like when you open it (figure 4c).

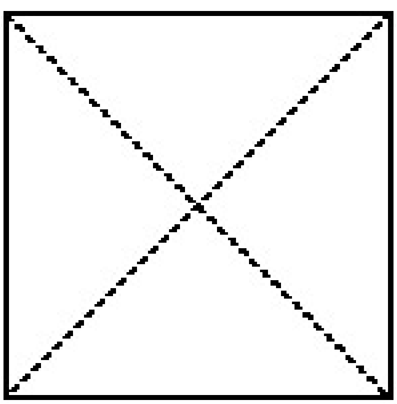


Fig. 4a. Square folded in 4 parts diagonally

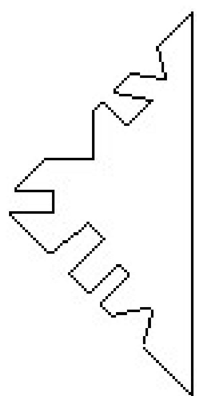


Fig. 4b. Cut along fold

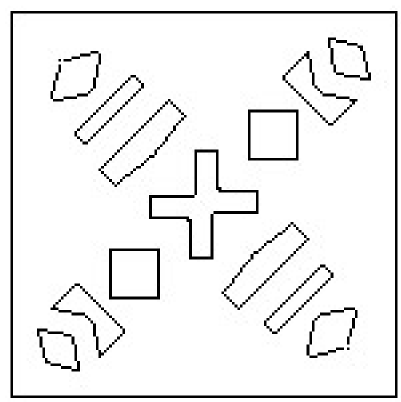


Fig. 4c. Two diagonal axes of symmetry

Patty Paper Activities

Activity 5. Four axes of symmetry.

Fold a square in four by making a mid parallel and a second perpendicular folds. Now fold the four-sheet square again along the diagonal that goes through the center of the original square (figure 5a). With the square folded, use scissors to cut triangles along the folds or along the borders (figure 5b). Predict what the paper will look like when you open it (figure 5c).

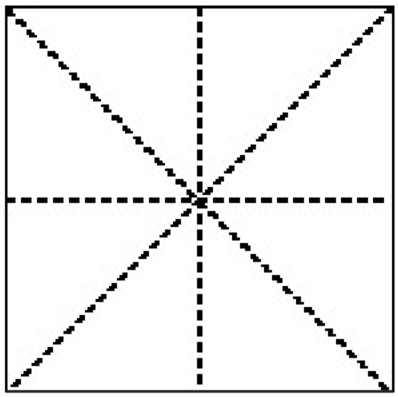


Fig. 5a. Square folded in 8 parts

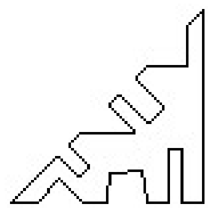


Fig. 5b. Cut along fold

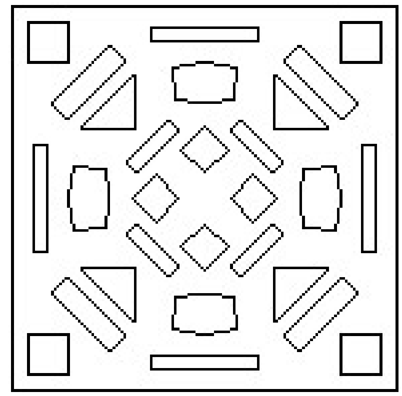
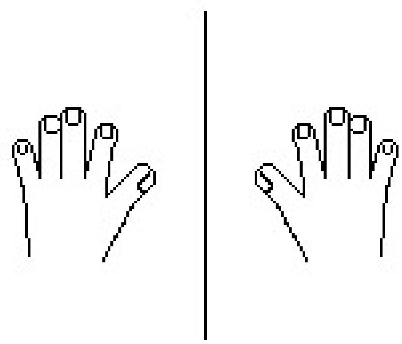


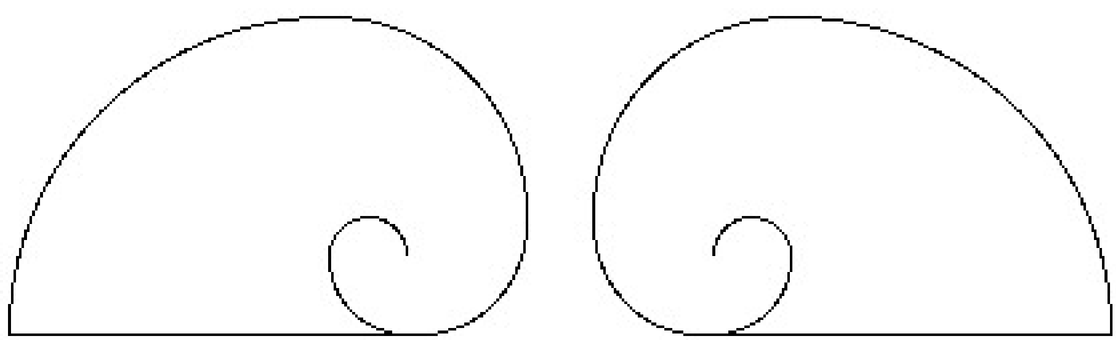
Fig. 5c. Four axes of symmetry

Symmetry and Asymmetry

Activity 1. Asymmetrical Objects

When we reflect an object in a mirror, we get another object that is very much like the original, yet in general not quite the same. Our left hand is the mirror image of our right hand. However a left glove does not fit in the right hand. Some objects are right-handed or left-handed, such as screws, shoes, gloves, some molecules, and some letters. By reflection in a mirror a right-handed object is turned into a left-handed object.



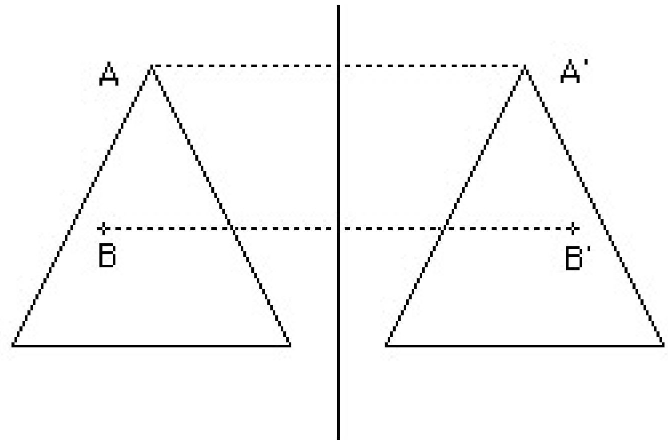


Symmetry and Asymmetry in Nature

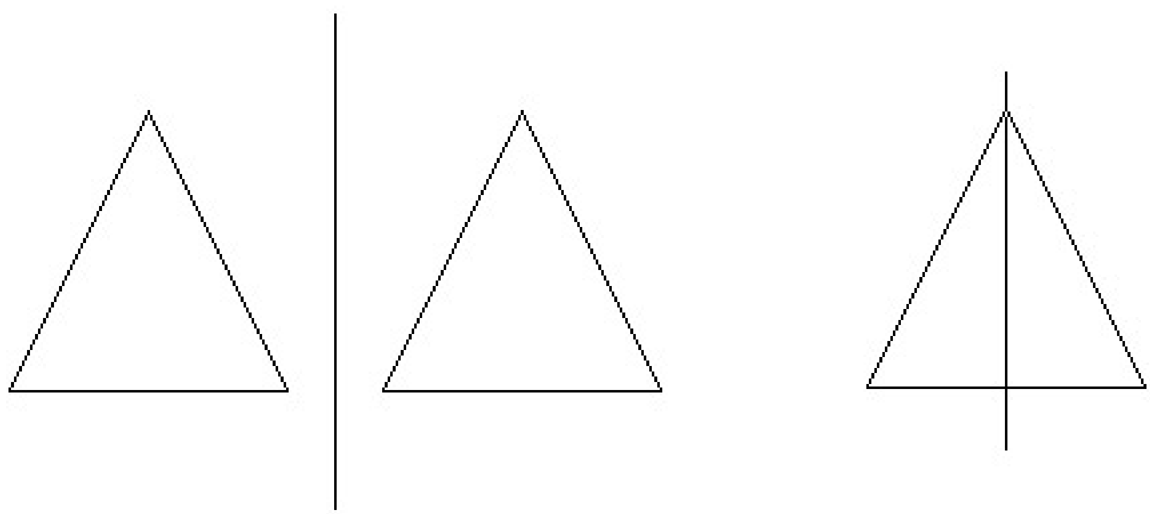
Activity 2. Mirror Symmetry

Objects that are symmetrical to each other

- The mirror image is obtained by reflecting each point on the other side of the mirror perpendicularly and at the same distance.



- Some objects are identical with their mirror images. For example an isosceles triangle and its image are exactly alike.

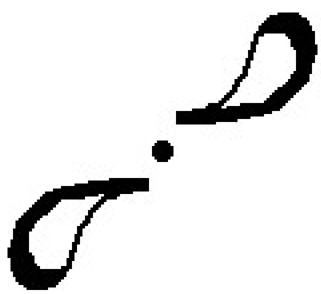


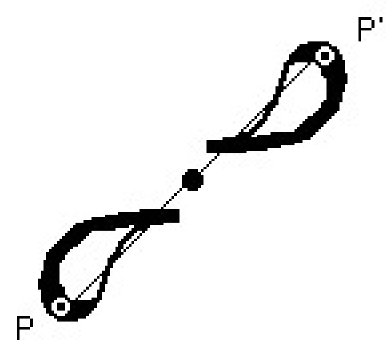
- In this case the objects have mirror symmetry. A two-dimensional figure has mirror symmetry if there is a bisecting line that divides the figure into mirror-image halves.
- A three dimensional figure has mirror symmetry if there is a bisecting plane that divides it into mirror halves.

Symmetry and Asymmetry in Nature

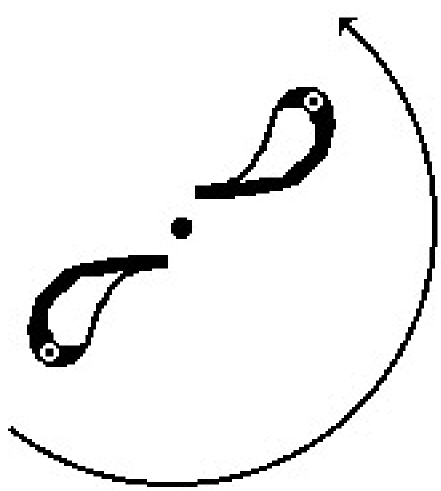
Activity 3. Point Symmetry

Reflecting through a fixed point produces another type of symmetry. To each point corresponds another point that is on a segment through the fixed point and is placed symmetrically on the other side of the fixed point.





We can see this type of symmetry also as a rotation of 180º.

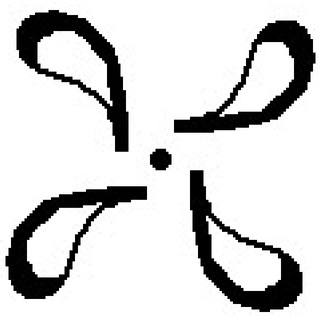


Reflecting a symmetrical object through it axis of symmetry leaves the figure invariant. Rotating 180˚ a figure that has point symmetry is also a transformation that leaves the figure invariant. We will use this idea of transformations and invariants to extend the concept of symmetry.

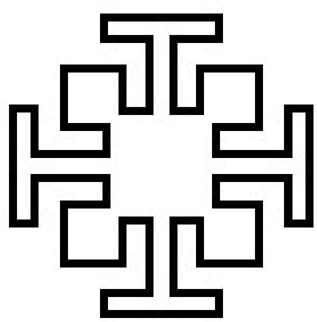
Symmetry and Asymmetry in Nature

Activity 4. Rotations

Some figures remain invariant under a rotation of 90˚ For example if we rotate a square 90º it will remain invariant. This figure also remains invariant with a 90˚ rotation.



Other figures are invariant under a rotation of 90˚ and have in addition two perpendicular mirror lines.

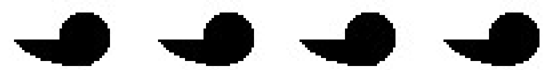


Symmetry and Asymmetry in Nature

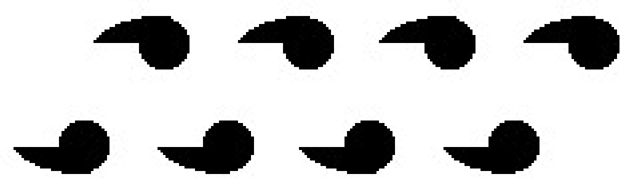
Activity 5. Translations

There are other transformations that can send an object onto itself. A repeating pattern, for example a strip in wall-paper, a strip in a carpet, or in a webbing can remain invariant under a translation.

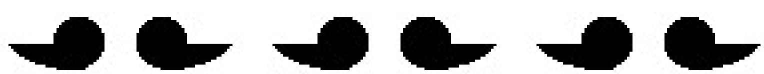
- There are seven types of strip patterns, each one a type of symmetry. The simplest pattern is an asymmetrical figure that repeats itself indefinitely.



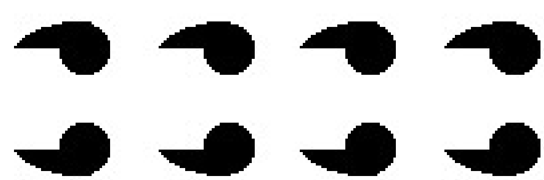
- Another pattern is glide reflection, where a figure and its glided reflection are repeated, like the footprints on the seashore.



- Some patterns have an inner mirror symmetry, a transversal axis, like the following example:



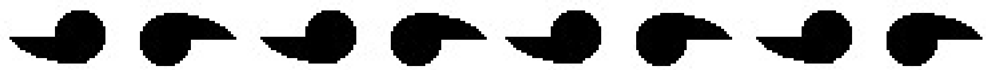
- Others have a longitudinal axis, like this one:



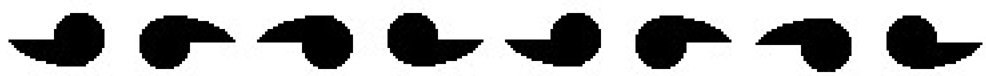
Symmetry and Asymmetry in Nature

Activity 5. continued

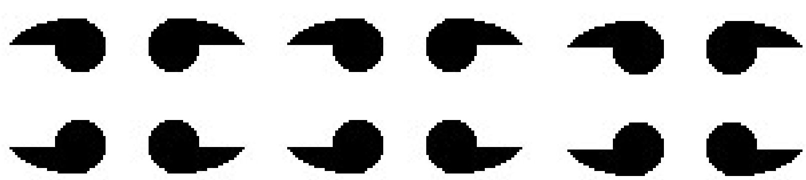
- In other patterns, a series of whirls (point symmetry, or 180º rotation) are repeated indefinitely.



- In the following example, a transversal axis of reflection is combined with point symmetry.



- Finally, the next pattern has both a transversal as well as a longitudinal axis of symmetry.



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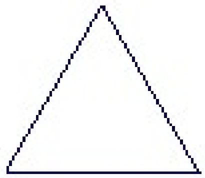
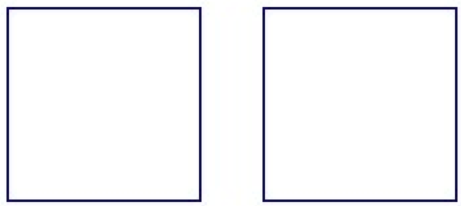
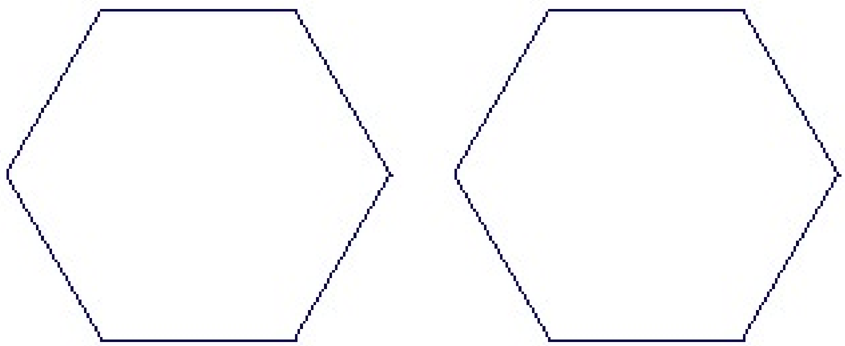
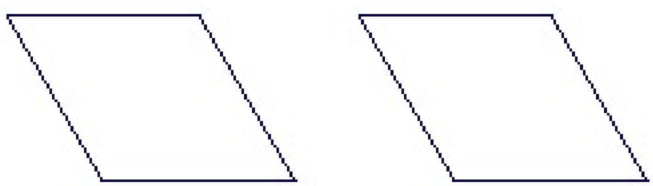
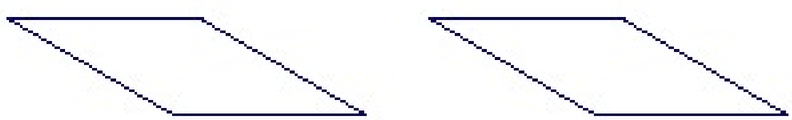
Exploring Symmetry with Pattern Blocks

Activity 1. Determine which of the pattern blocks have mirror symmetry. Use a Mira to verify that the shape is indeed symmetric. If a particular piece has more than one kind of axis of symmetry note so. Trace the axes of symmetry.

Shape	Axes of symmetry
Equilateral triangle	
Square	
Hexagon	
Trapezoid	
Blue rhombus	
Tan rhombus	

Exploring Symmetry with Pattern Blocks

Activity 2. Combine several pattern blocks to design a shape that is symmetrical (mirror symmetry). Use a Mira if necessary to verify that your design is indeed symmetrical.

Activity 3. Construct a design that has point symmetry. Identify the center of symmetry. Trace your design on paper. Trace lines across the point of symmetry to identify points on your design that are symmetrical with respect to the center of symmetry.

Activity 4. Construct a design with pattern blocks that has point symmetry but does not have mirror symmetry.

Activity 5. Construct a design with pattern blocks that has both point symmetry and mirror symmetry.

Activity 6. Verify by rotating that a shape with point symmetry also has a rotational symmetry of 180° around the center of symmetry.

Activity 7. Verify with several shapes that the following two conditions are equivalent. That is, a shapes that satisfies condition a) also satisfies condition b) and vice versa.

a) A shape has point symmetry and one axis of mirror symmetry.
b) A shape has two perpendicular axes of mirror symmetry.

Activity 8. Identify which of the pattern block pieces have point symmetry.

Activity 9. Verify that the square has rotational symmetry of 90° with respect to its center.

Activity 10. Construct a design that has rotational symmetry of 90°. Identify the center of rotation.

Activity 11. Construct a design with pattern blocks that has rotational symmetry of 90° but does not have mirror symmetry.

Activity 12. Construct a design with pattern blocks that has both rotational symmetry of 90° and has also mirror symmetry. Identify all the symmetry axes of your design.

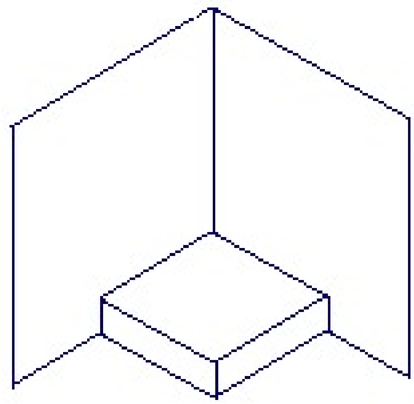
Activity 13. *What are the rotational symmetries of the equilateral triangle?*

Activity 14. Identify the rotational symmetries of the regular hexagon.

Activity 15. *Are there any other pattern block pieces that have rotational symmetry of any kind?*

Hinged Mirrors and Pattern Blocks

1. Place a square between the two hinged mirrors so that they form a 90˚ angle.
   How many squares do you see? (Including the original.)



Square pattern block between hinged mirrors.
Reflections not shown.

2. Place a triangle between the two hinged mirrors so that they form a 60˚ angle.
   How many triangles do you see?
3. Place a hexagon between the two hinged mirrors so that they form a 120˚ angle.
   How many hexagons do you see?
4. Place the tan rhombus between the two mirrors so that they form a 30˚ angle.
   How many rhombuses do you see?
5. Fill the table below.

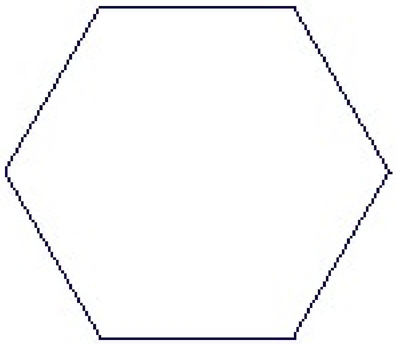
Shape	Angle between mirrors	Number of shapes
Tan rhombus		
Equilateral triangle		
Square		
Regular hexagon		

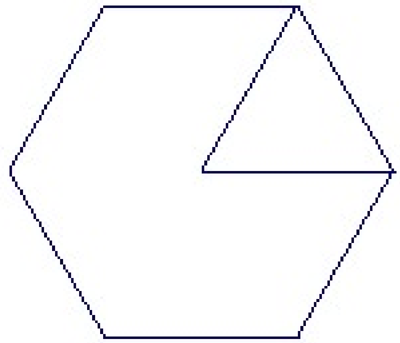
   What do you notice about the values in the table?
6. Place the other pattern block shapes between the mirrors in different positions. For example, place the blue rhombus to form first a 60˚ angle between the mirrors, and then to form a 120˚ angle.
   Is the number of shapes consistent with the table above?
7. Express the relation between the size of the angle and the number of shapes with an equation.
8. Now place the tan rhombus between the two mirrors and open the angle so that you see exactly five rhombuses (including the original).
   What is the angle formed by the mirrors?

Simple Kaleidoscopes

Activity 1. Hexagon

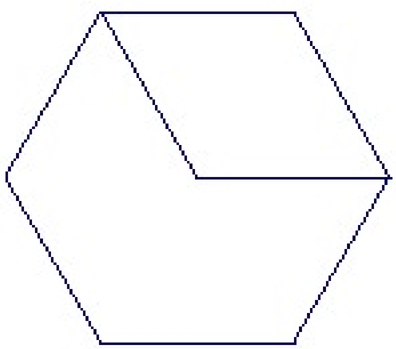
Draw all the mirror lines of the hexagon.

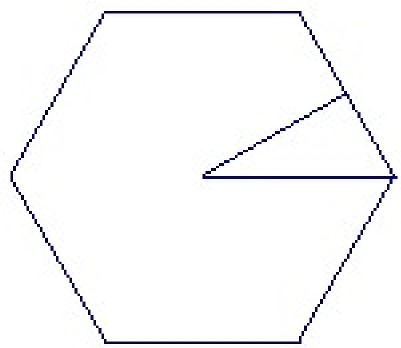




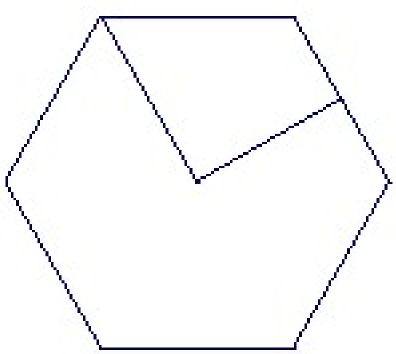
a) Place mirrors along two mirror lines, with the mirrors meeting at the center of the hexagon. If you look between the mirrors you should see the entire hexagon. Place a few tiny objects between the two mirrors. If you look you will see a pattern that is kaleidoscopic.

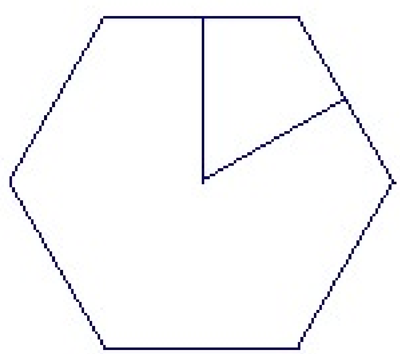
b) Place hinged mirrors along the lines indicated.





c) See whether you can reconstruct the whole hexagon by placing the hinged mirrors along the indicated lines.



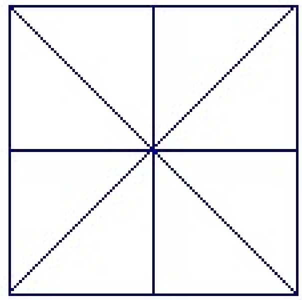


d) *What can you say about the angles for which you can reconstruct the entire hexagon with the hinged mirrors?*

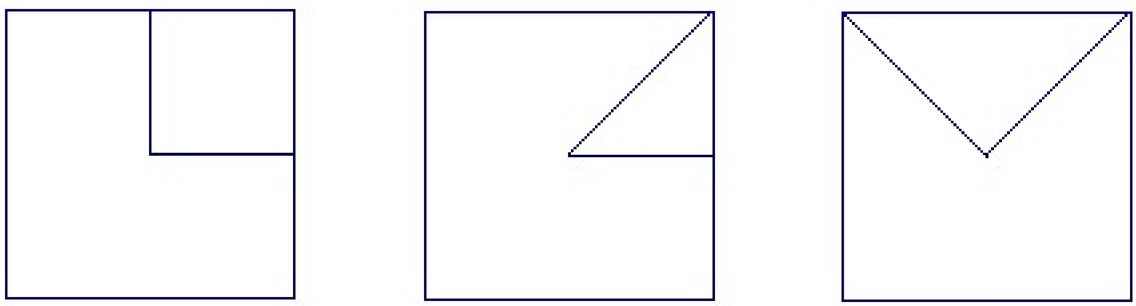
Simple Kaleidoscopes

Activity 2. Square.

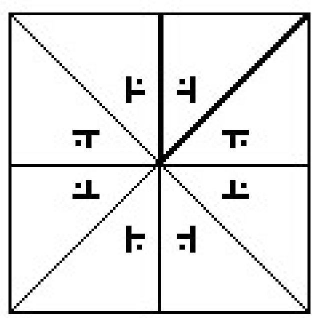
Construct a square with all its mirror lines on it.



a) Place the hinged mirrors along two mirror lines, with the mirrors meeting in the center of the square. If you look between the mirrors you should see the entire square.



b) *What can you say about the angles that will form a complete square when reflected?*



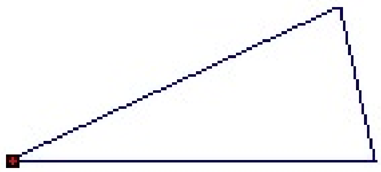
c) Place a few tiny objects between the two mirrors. If you look you will see a pattern that is kaleidoscopic.

Explain why the kaleidoscopic image is formed.

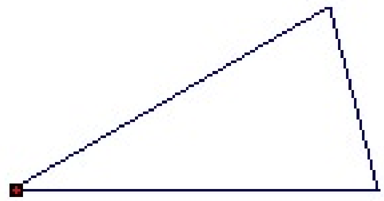
Simple Kaleidoscopes

Activity 3

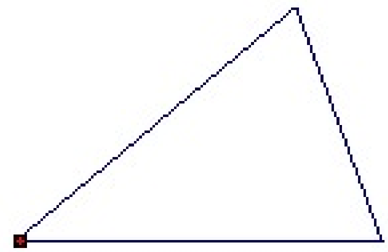
Place mirrors along the lines that meet at the marked vertex of the triangle. *Which of the angles produce a regular polygon?* You may also try your own angles. Express the condition satisfied by angles that give rise to a regular polygon in your own words.



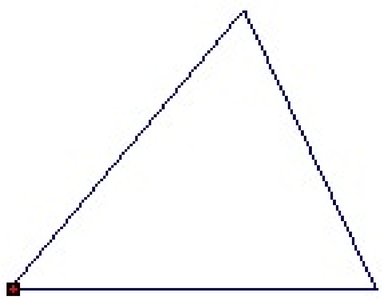
25°



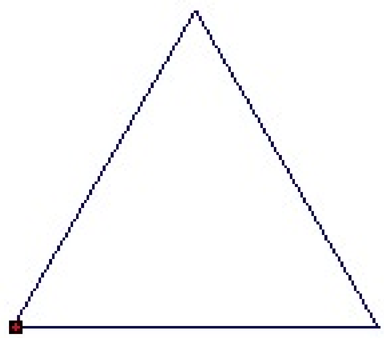
30°



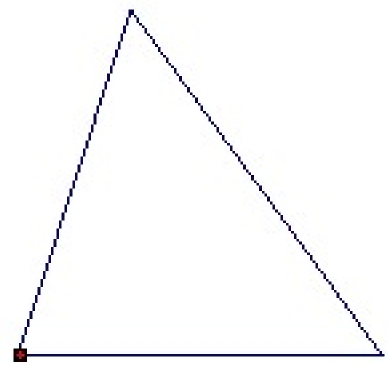
40°



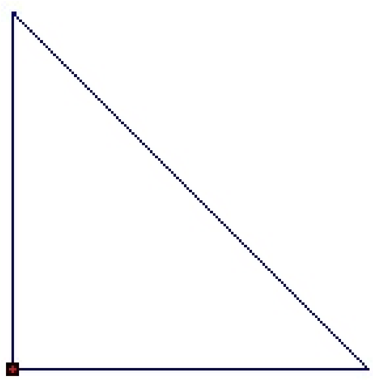
50°



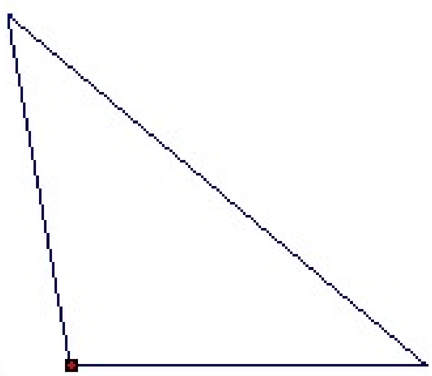
60°



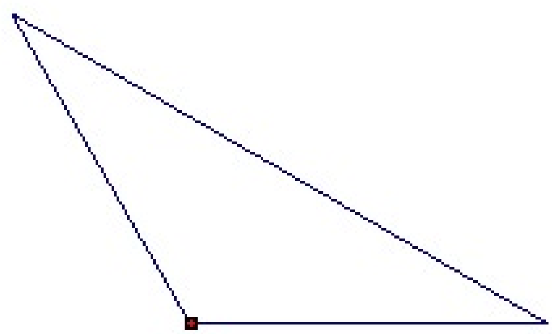
72°



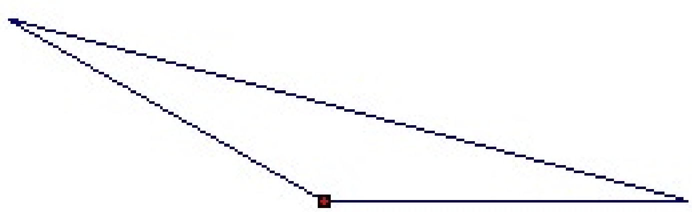
90°



100°



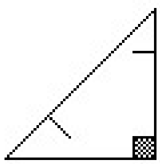
120°



150°

Three Mirror Kaleidoscopes

1. The kaleidoscope made of two mirrors generates finite patterns with the symmetries of the regular polygons.
2. There are also kaleidoscopes whose patterns appear to be infinite in extent. Three mirrors joined to form a prism are used.
   - Use three mirrors to form a prism whose base is an equilateral triangle.
   - Place colored patterns in the base of the kaleidoscope.
   - *Can the regular tessellations be generated by placing patterns in an equilateral 3 mirror kaleidoscope? If so, show the patterns.*
   - *How many of the semi regular tessellations can be generated by placing patterns in an equilateral three mirror kaleidoscope? Show the patterns.*
3. Kaleidoscopes that produce infinite patterns can also be obtained with triangles other than the equilateral.
   - The angles for one or these kaleidoscopes are 30°, 60°, and 90°. For the other the angles are 90°, 45°, and 45°.
   - The tessellation in Figure 1 was generated by placing the indicated pattern in the base of the kaleidoscope with angles 90°, 45°, 45°.



Basic pattern

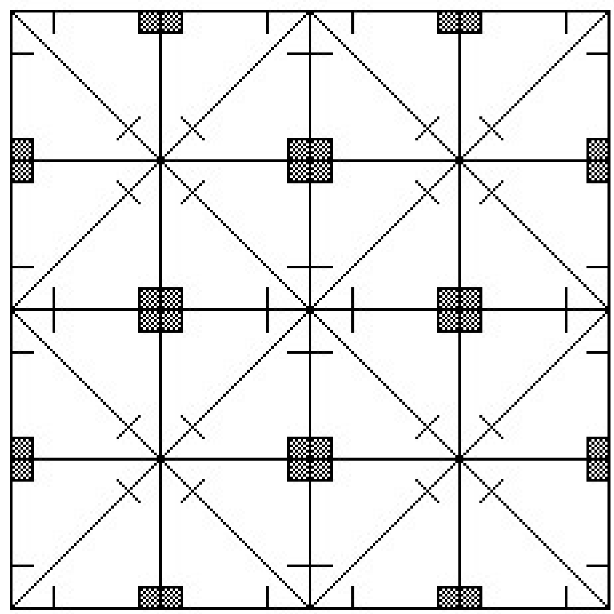


Figure 1

4. Each of the patterns in Figure 2 when placed in the base of a 90-45-45 three-mirror kaleidoscope generates a tessellation.

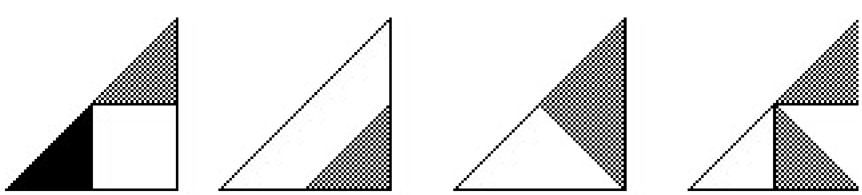


Figure 2

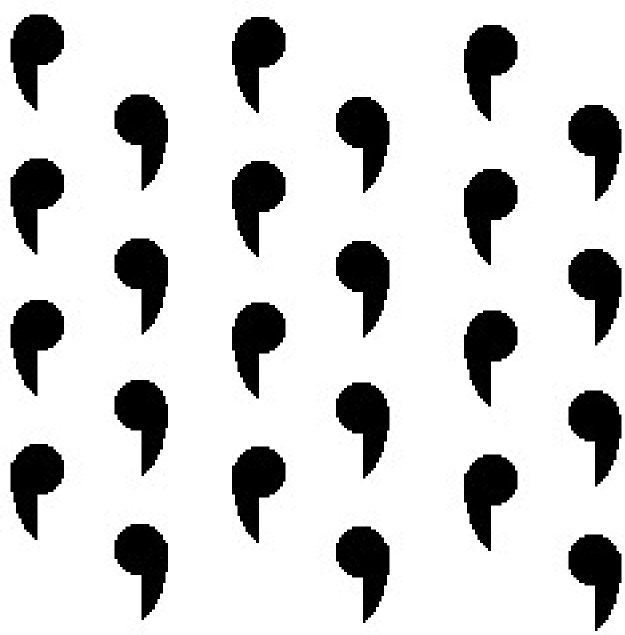
For each pattern guess what the tessellation will look like. Check your guess with a kaleidoscope.

Symmetries in the Plane

Supplementary material for groups that move at a faster pace than other groups in the class

On the plane patterns can be classified into 17 groups of symmetries. The simplest pattern consists in two non parallel translations.

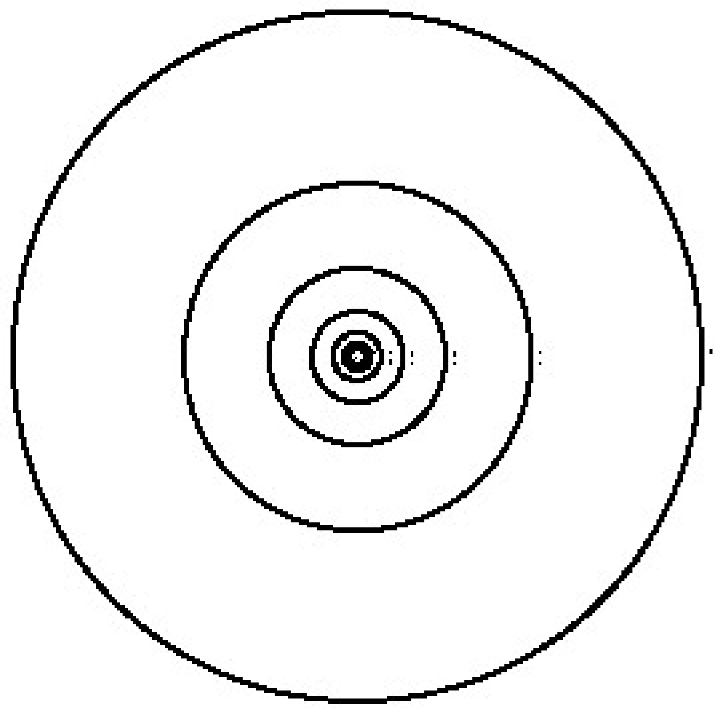
Another common pattern consists in two parallel glided reflections.



Arab patterns, Escher drawings illustrate and explore these patterns. It is common to find in the art of cultures that the designs cover all 17 groups.

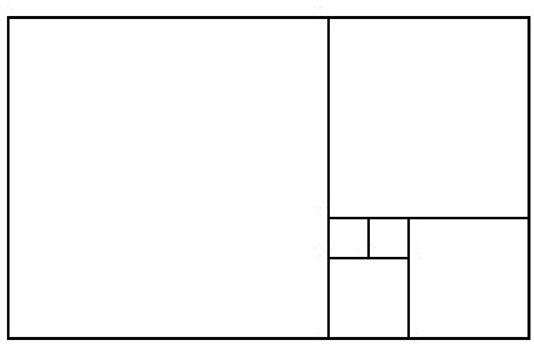
Similarity

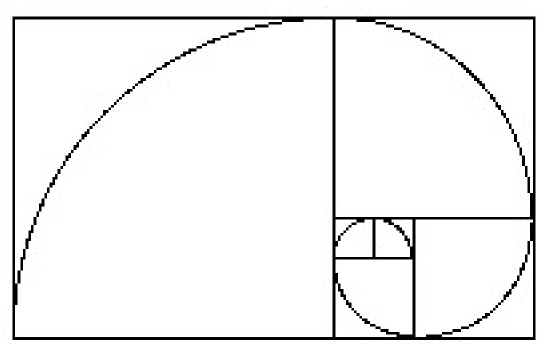
Not only rigid transformation is capable of transforming an object into itself.



For instance, a set (infinite) of concentric circles, each one with a radius twice as big as the preceding, can be transformed into itself by a dilation or a contraction by a factor of 2.

We can combine different types of these transformations, for example in a snail, a chamber can be transformed into an other by means of a dilation and a rotation. This fact allows the organism to grow while keeping the shape of the chamber the same.





Such a spiral remains invariant if we combine a rotation with a dilation in the proper way.

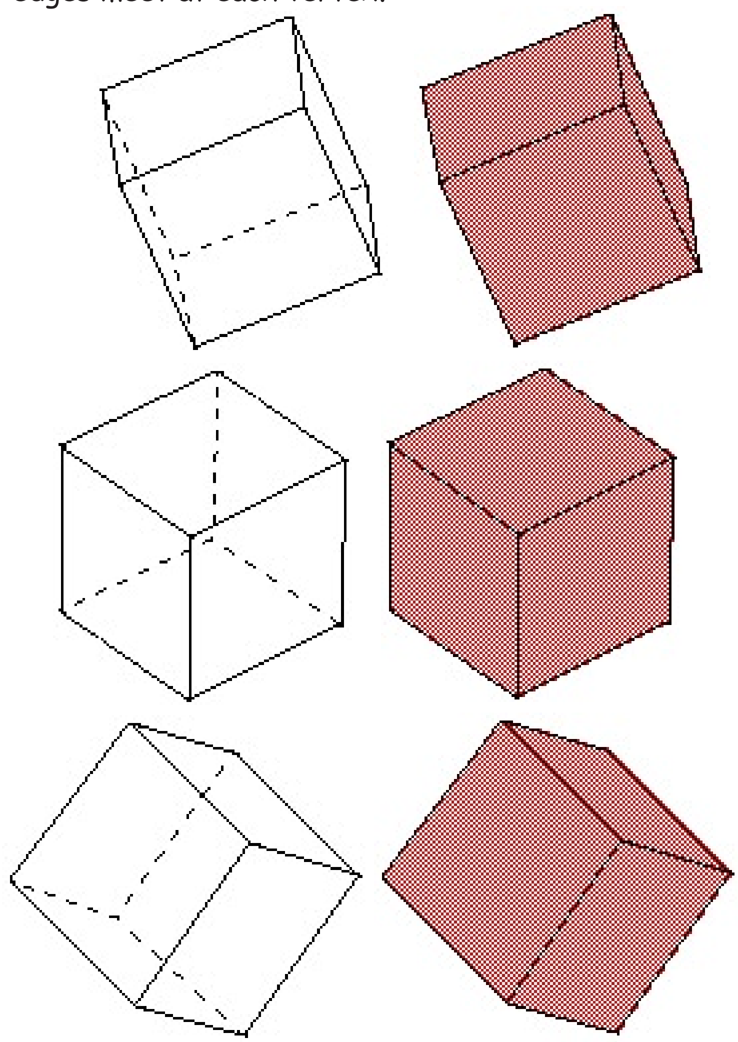
Regular Polyhedra Definition

Description and Definition of Regular Polyhedra

Regular polyhedra are characterized by two properties:
1) Their faces are congruent, convex, regular polygons.
2) The same number of polygons meets at each vertex.

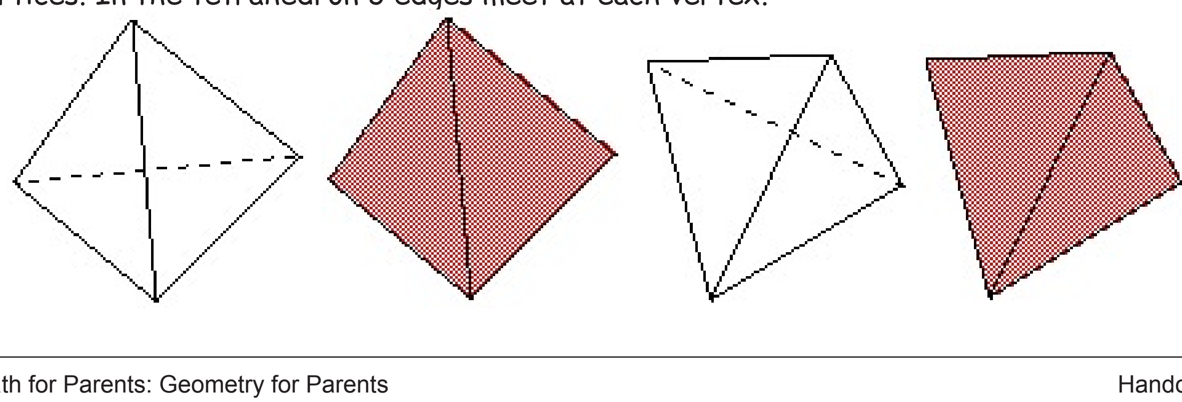
The Cube

The most familiar of the regular polyhedra is the cube. It has 6 square faces, 12 edges, and 8 vertices. In the cube, 3 edges meet at each vertex.



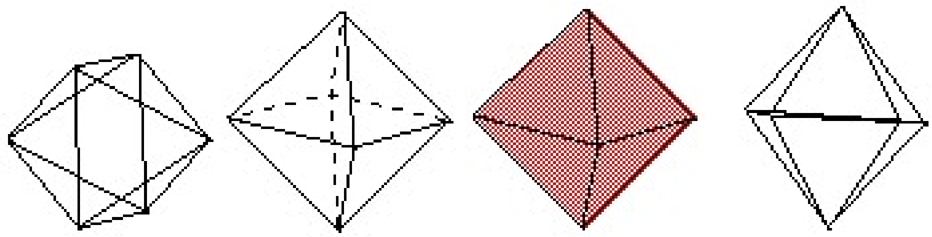
The Tetrahedron

The tetrahedron is the simplest of the regular polyhedra. It has 4 triangular faces, 6 edges, and 4 vertices. In the tetrahedron 3 edges meet at each vertex.

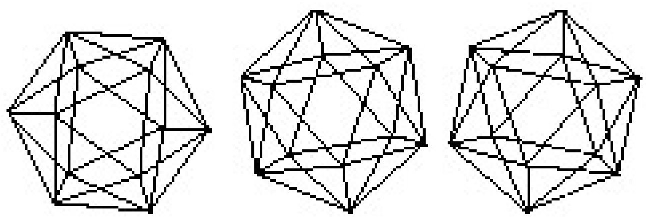


Regular Polyhedra Definition

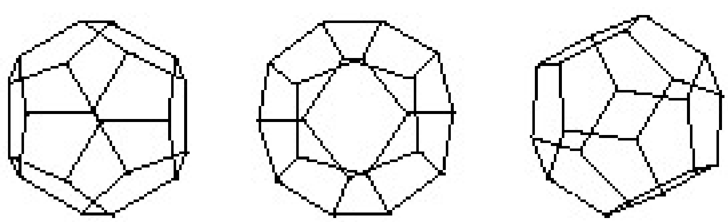
The Octahedron
The octahedron has 8 triangular faces, 12 edges, and 6 vertices. In the octahedron, 4 edges meet at each vertex.



The Icosahedron
The icosahedron has 20 triangular faces, 30 edges, and 12 vertices. In the icosahedron, 5 edges meet at each vertex.



The Dodecahedron
The dodecahedron has 12 pentagonal faces, 30 edges, and 20 vertices. In the dodecahedron 3 edges meet at each vertex.

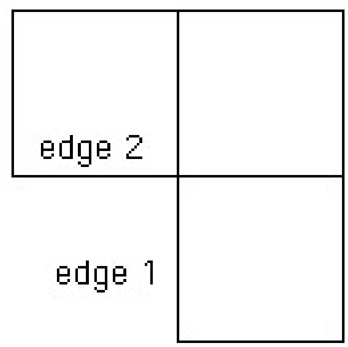


Building Regular Polyhedra

There are commercially available sets that allow for easy construction of polyhedra. Some of these are the frames made of plastic or cardboard sets. Polyhedra can also be built from paper using the **Nets for Building Polyhedra** handout.

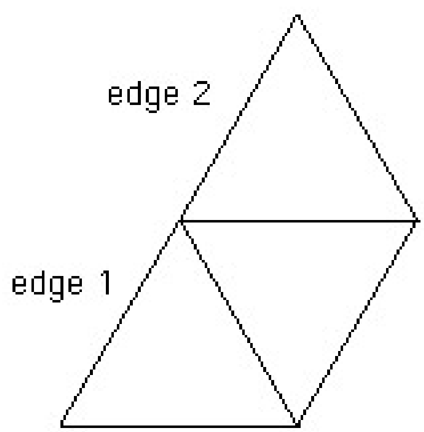
Activity 1. The Cube

Join three squares so that they share a vertex. Now join edge 1 with edge 2. Finish the cube with three more squares.



Activity 2. The Tetrahedron

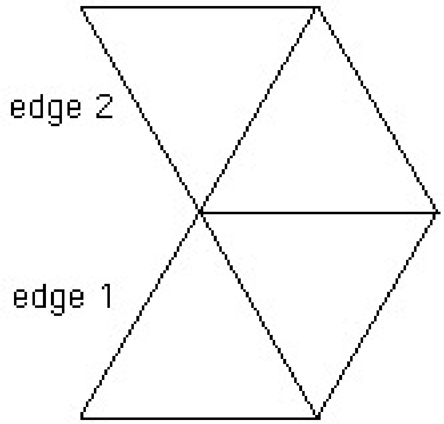
Join three triangles so that they share one vertex. Now join edge 1 with edge 2. Complete the tetrahedron with one more triangle.



Building Regular Polyhedra

Activity 3. The Octahedron

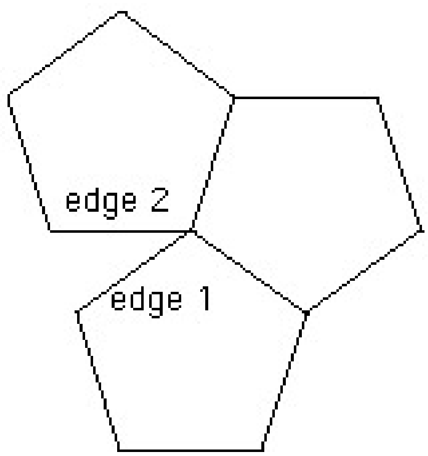
Join four triangles so that they share one vertex. Now join edge 1 with edge 2. Complete the octahedron by adding four more triangles, so that at each vertex you have four triangles meeting.



Activity 4. The Dodecahedron

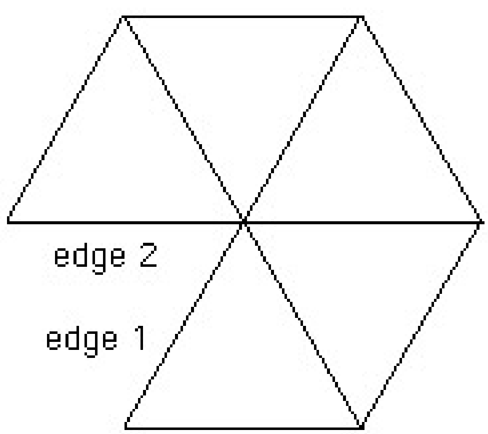
Join three pentagons so that they share a vertex.

Now join edge 1 with edge 2. You will need nine more pentagons to finish the dodecahedron. Three pentagons meet at each vertex.



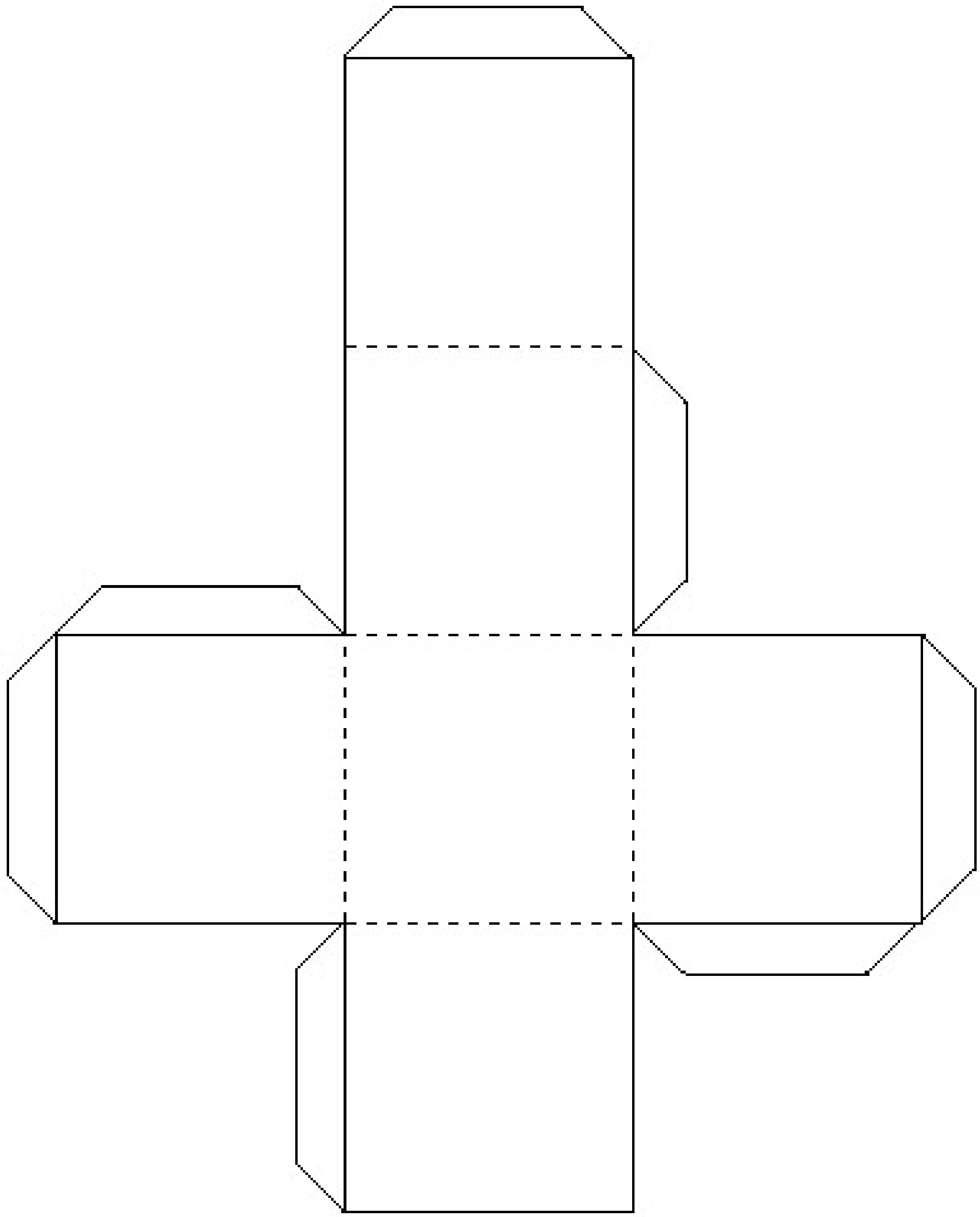
Activity 5. The Icosahedron

Join five triangles so that they share one vertex. Now join edge 1 with edge 2. To finish the icosahedron you will need another "cap" like this, and then 10 more triangles forming a "belt" between them. The icosahedron has 20 triangular faces, where 5 triangles meet at each vertex.



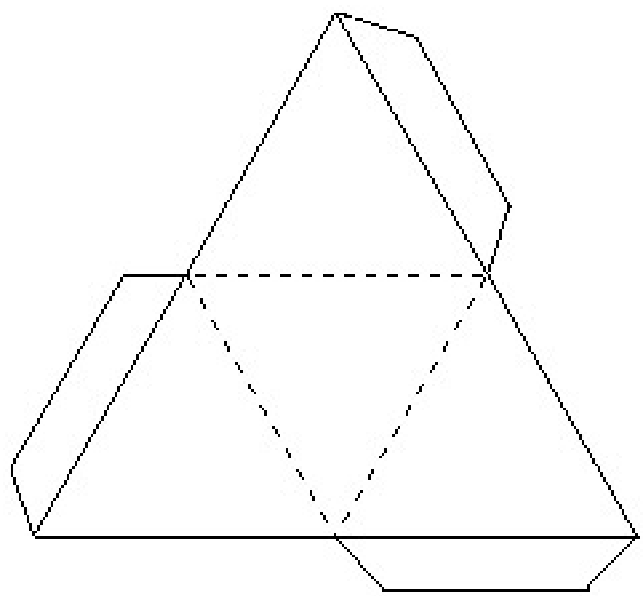
Nets for Building Polyhedra

Net Cube



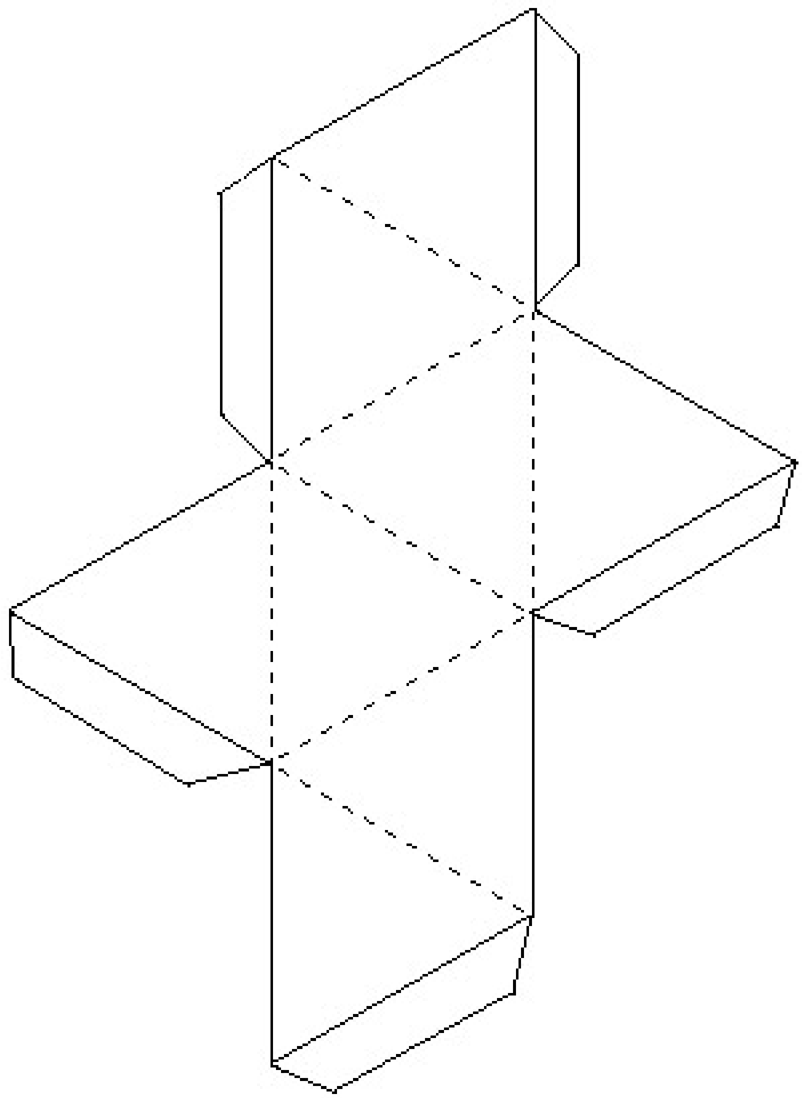
Nets for Building Polyhedra

Net Tetrahedron



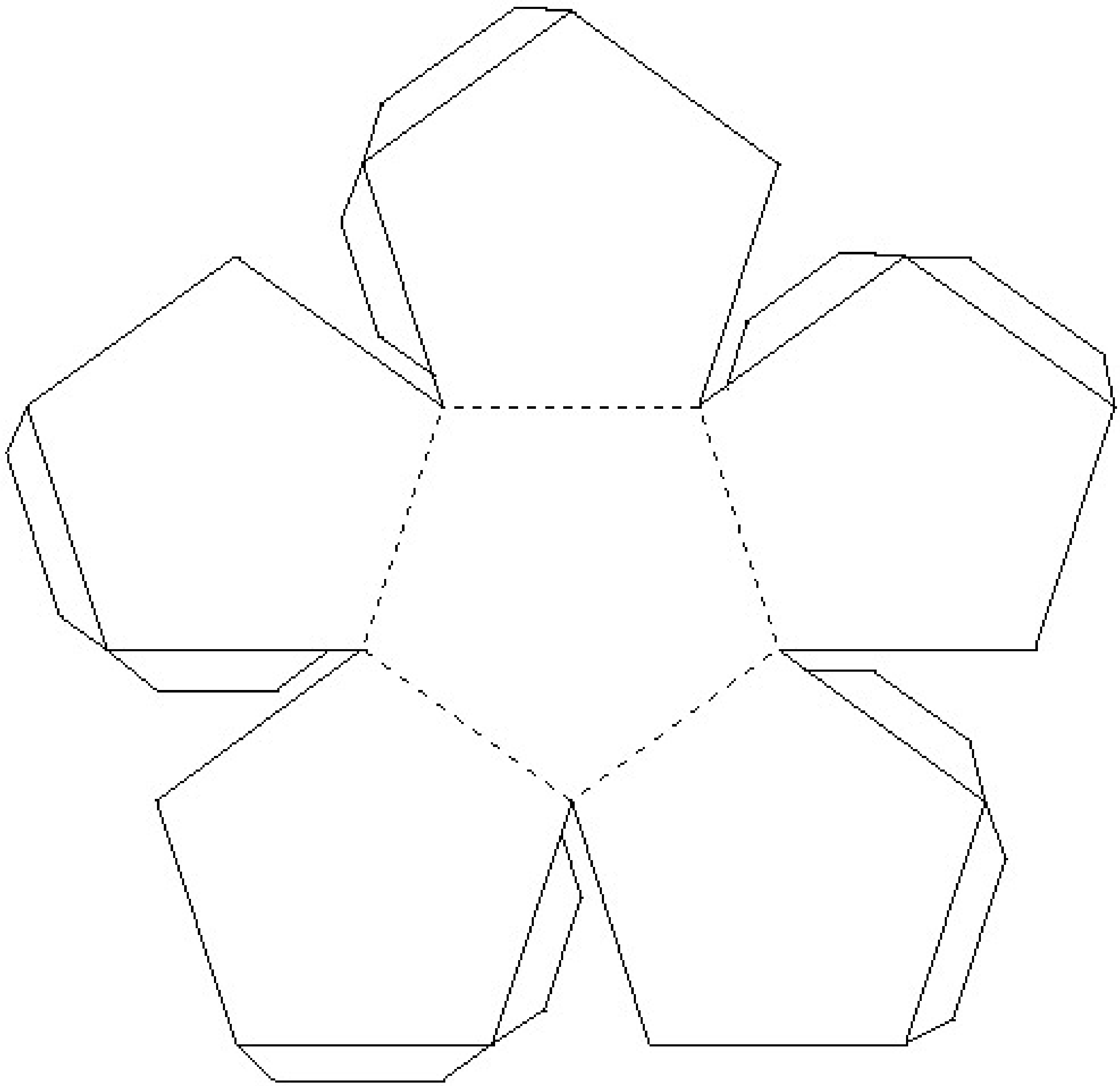
Nets for Building Polyhedra

Net Octahedron



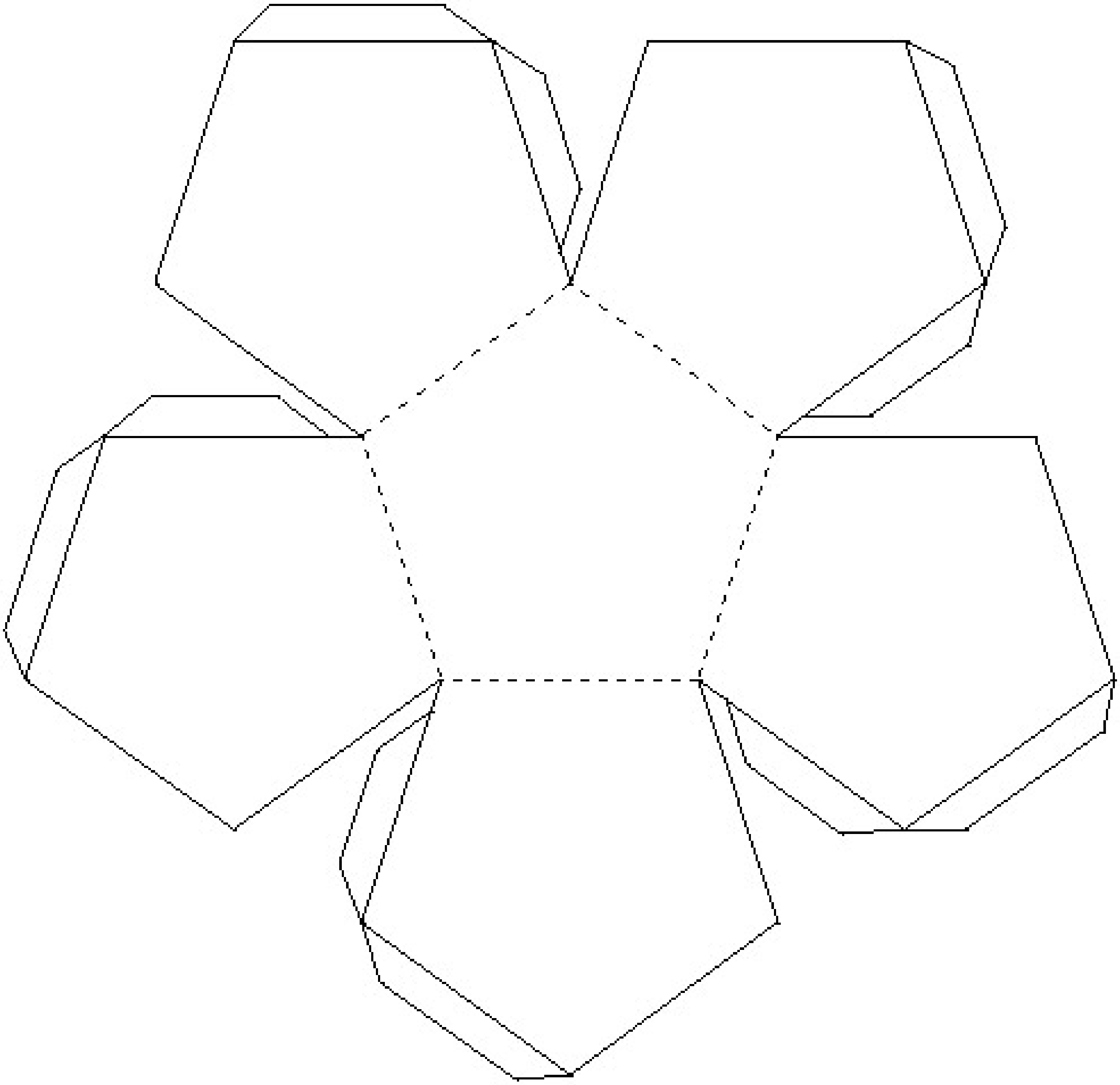
Nets for Building Polyhedra

Net Dodecahedron - Part 1



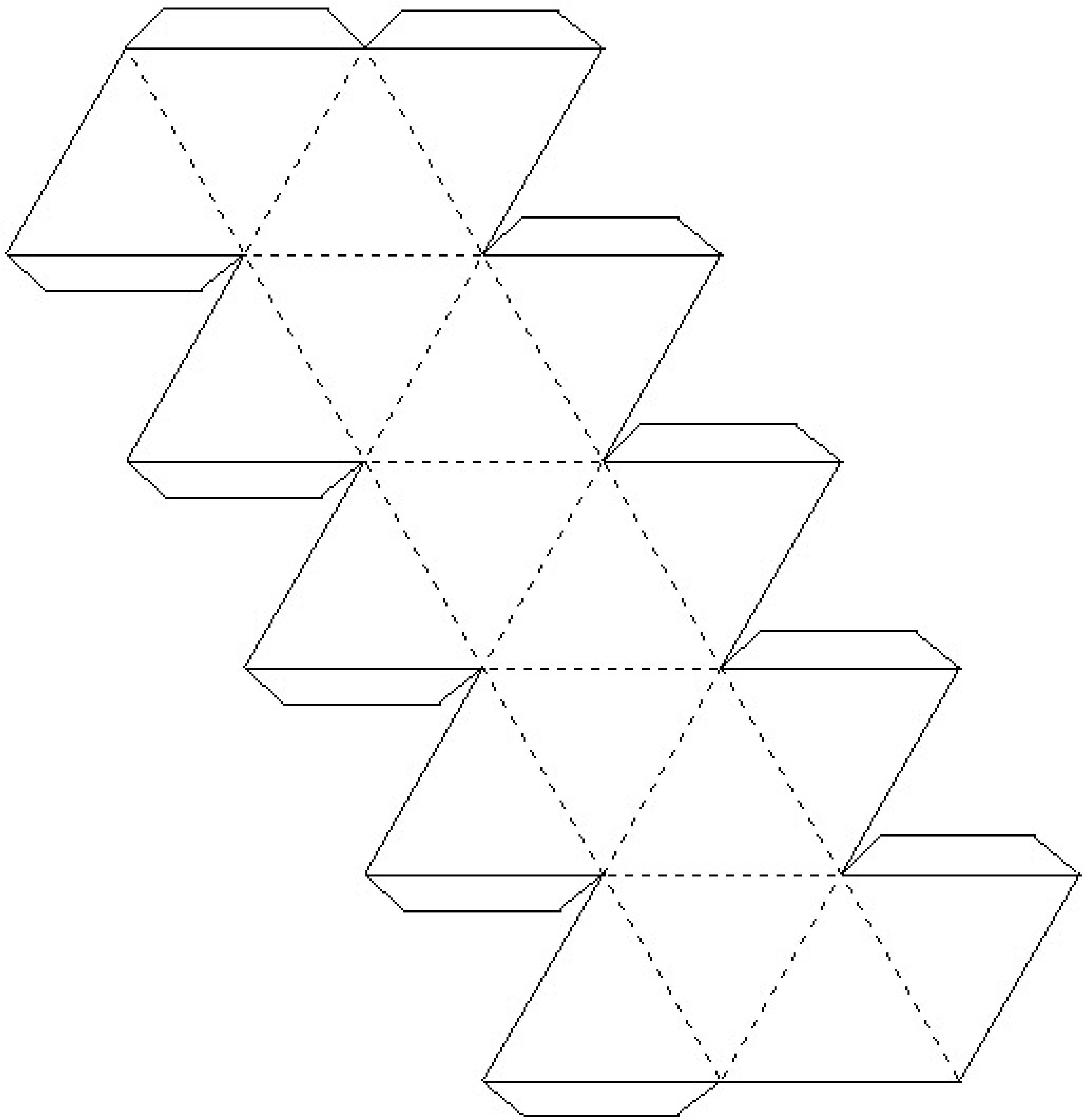
Nets for Building Polyhedra

Net Dodecahedron - Part 2



Nets for Building Polyhedra

Net Icosahedron



Only Five Regular Polyhedra

Activity 1. Counting the Faces, Edges, and Vertices of Regular Polyhedra

Why are there only five regular polyhedra?
At least three polygons are required to create a polyhedral vertex, and the number that can fit together at that vertex depends on the polygonal angle. The sum of the polygonal angles at any vertex must be less than 360º, because otherwise the configuration of polygons could not be folded up.

- If we use equilateral triangles we can have: 3 triangles, 4 triangles or 5 triangles meeting at a vertex (six triangles form a flat configuration, because the sum of the angles meeting at one vertex is 360º.
- If we use squares we can have 3 squares only meeting at a vertex (four squares form a flat configuration)
- If we use regular pentagons we can have only 3 meeting at a vertex (the sum of the angles of four pentagon meeting at a vertex is 4 x 108o, which is greater than 360º.
- We cannot use 3 hexagons, because they form a flat configuration.
- The sum of three angles of a regular heptagon or polygon with more than 6 sides is greater than 360º.
- So the only possibilities are 3 triangles, 4 triangles or 5 triangles meeting at a vertex (tetrahedron, octahedron, icosahedron); 3 squares meeting at a vertex (cube); or 3 regular pentagons meeting at a vertex (dodecahedron).

How to count the edges of a regular polyhedron
The icosahedron has 20 triangular faces, that is, each face is formed by 3 edges, but each edge is shared by 2 faces. So the number of edges is 20 x 3 / 2 = 30

The dodecahedron has 12 pentagonal faces, that is, each face is formed by 5 edges, but each edge is shared by 2 faces. The number of edges is 12 x 5 / 2 = 30

Only Five Regular Polyhedra

Activity 2. Euler's Formula for Polyhedra

Table for Counting Regular Polyhedra

	Number of faces	Number of vertices	Number of edges	
Tetrahedron				
Cube				
Octahedron				
Dodecahedron				
Icosahedron				

Additional Activities

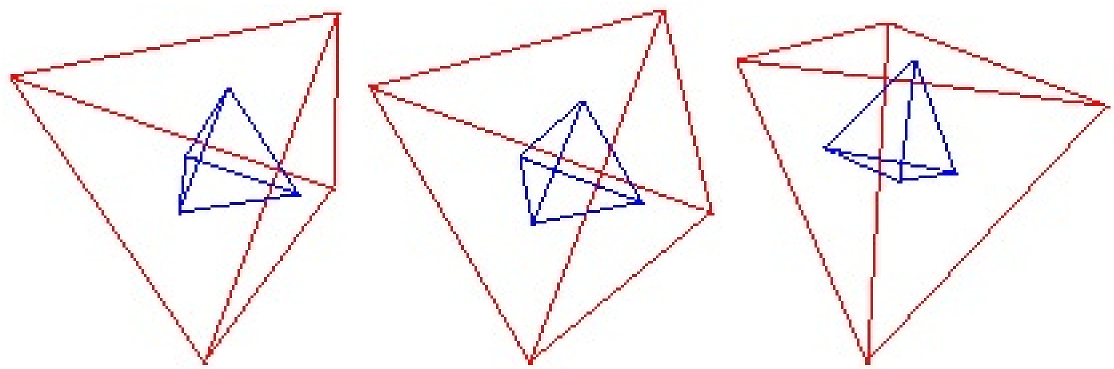
- Five **Nets for Building Polyhedra** polyhedra are supplied.
- Cut them and fold them up to make the models.
- Design another net for the tetrahedron.

Only Five Regular Polyhedra

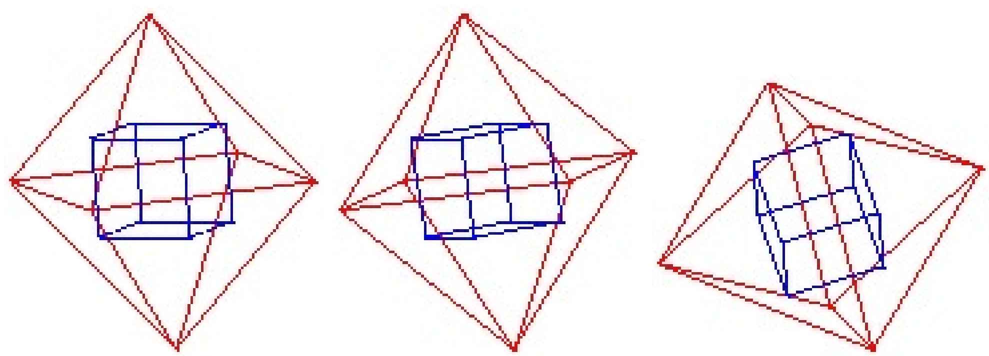
Activity 3. Duality

Dual polyhedra have a special relationship. The number of vertices in one is the number of faces in the other, and vice versa. They have the same number of edges. Dual polyhedra can be inscribed in another.

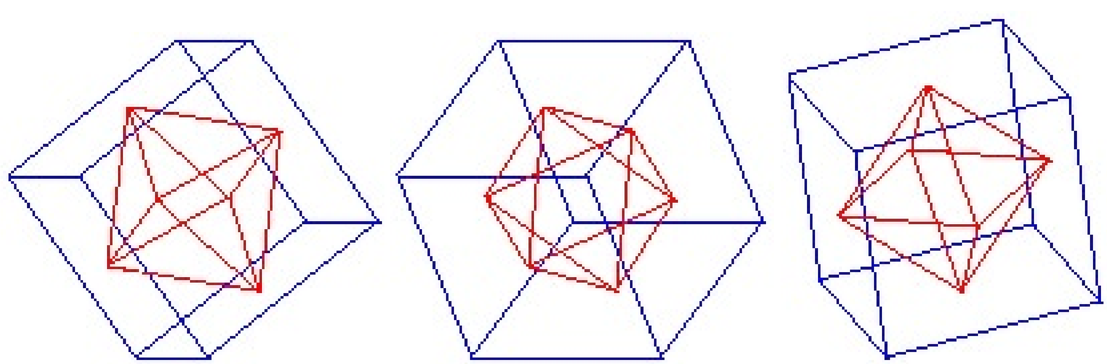
A tetrahedron is formed when joining the midpoints of the faces of a tetrahedron. That is, the tetrahedron is its own dual.



Inscribe a cube in an octahedron connecting the midpoints of adjacent faces.



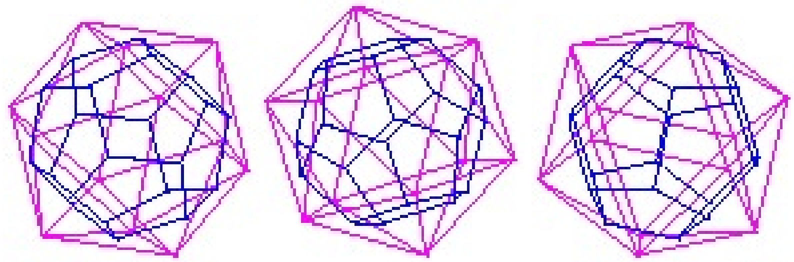
We can also inscribe an octahedron in a cube connecting the midpoints of adjacent faces.



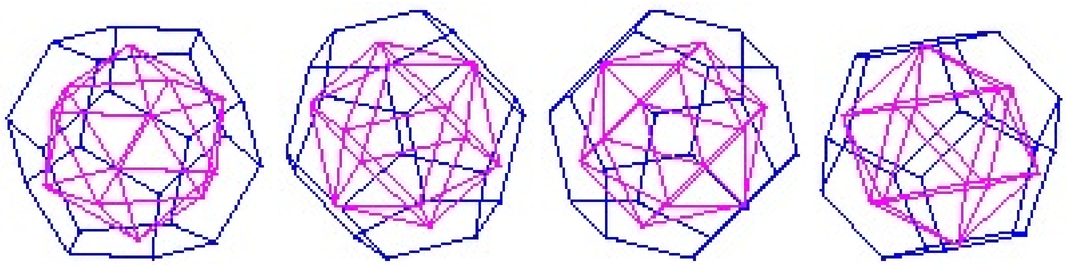
Only Five Regular Polyhedra

Activity 3. Continued

Dodecahedron in Icosahedron



An icosahedron can be inscribed in its dual, the dodecahedron.



Nets for the Cube

A cube can be made out of a single piece of paper cut in the shape of a cross. The cross is made of six squares, so that when you fold it up, a cube is formed (see figure 1). The cross is a net for a cube.

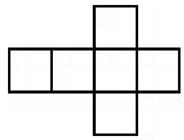


Figure 1

Any flat configuration of polygons, joined along whole edges, which can be folded up to make a polyhedron is called a net for the polyhedron. In general, a polyhedron can have many different nets, although of course all of them have the same kinds and numbers of polygons. Figure 2 shows three more nets for a cube.

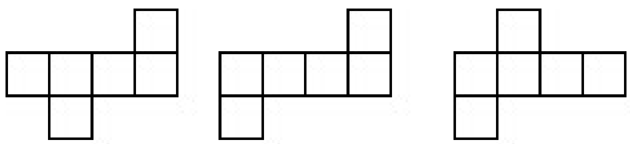


Figure 2

There are many ways to design a net for a cube. The rule is that it must be possible to cut the net out in one piece, so in a net each square must be joined to another square along a common edge. Not every configuration of six squares will fold up into a cube.
Which of the following are nets for a cube?

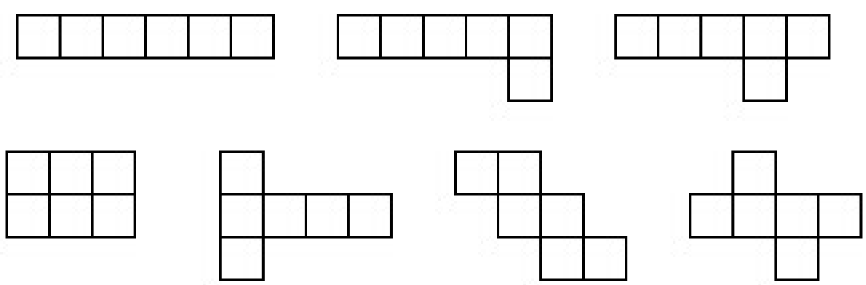
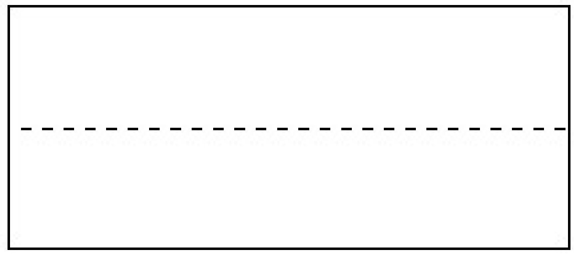


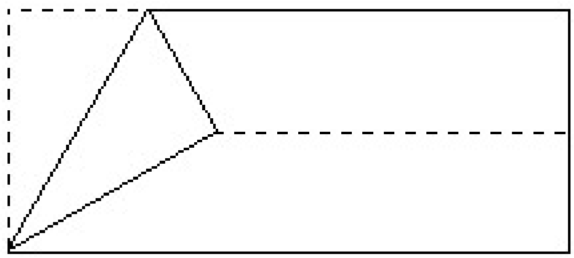
Figure 3

The Dollar Bill Tetrahedron

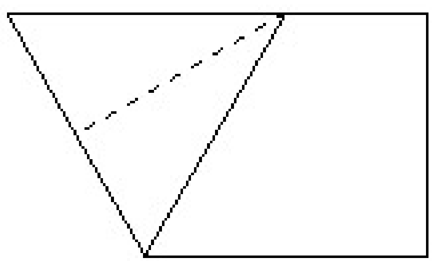
1) Fold a crisp dollar bill in half along the long mid parallel.



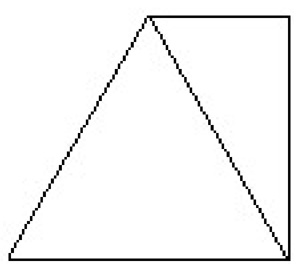
2) Fold one corner of the dollar bill onto the mid parallel, so that the crease passes through the adjacent corner to form a 60˚ angle.



3) Fold the bill to bisect the 120˚ angle to form another 60˚ angle and an equilateral triangle.

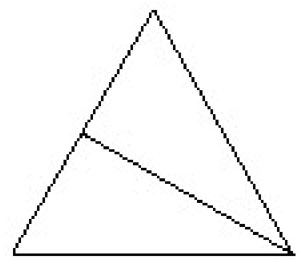


4) Fold again to bisect the new 120˚ angle to form another equilateral triangle.

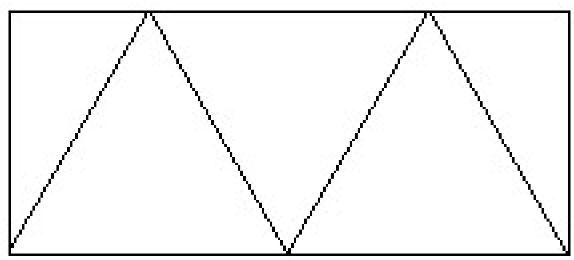


The Dollar Bill Tetrahedron

5) Fold the remaining flap.



6) When you open the dollar you should see the zigzag pattern shown below.



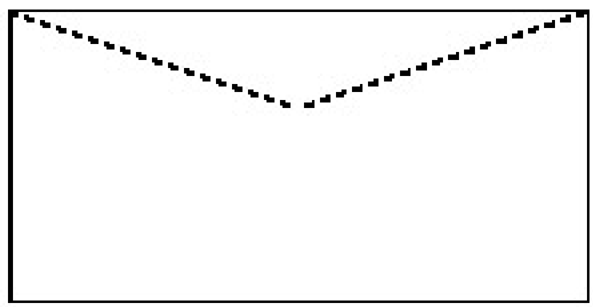
7) Place the flaps side by side and fold to form a tetrahedron.

- *What is the ratio of the two sides of a rectangle that folds exactly into a tetrahedron (the way the dollar bill was folded)?*
- *How close is the dollar bill to that shape?*

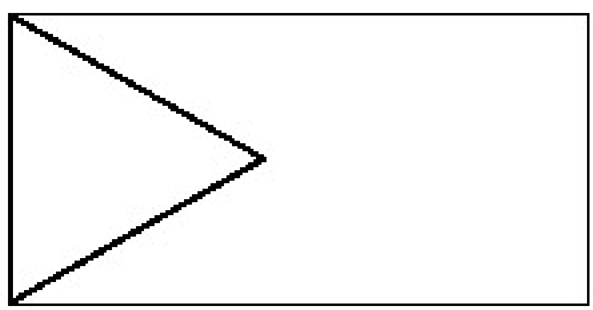
Tetrahedron from Envelope

Supplementary material for groups that move at a faster pace than other groups in the class.

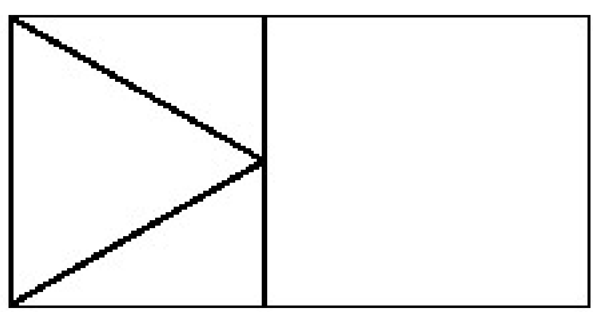
1) A tetrahedron from a closed envelope



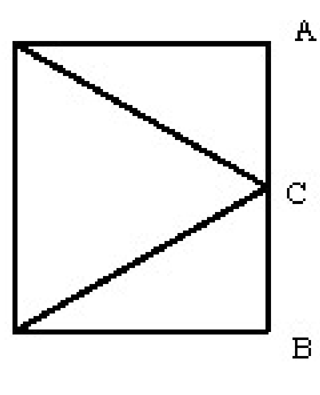
2) On one side of the closed envelope trace an equilateral triangle with its side equal to the shorter side of the envelope.



3) Trace a parallel line to the shorter side through the vertex of the equilateral triangle. Cut along the line.



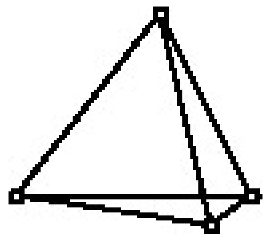
4) Fold along the sides of the triangle. Push A towards B, and separate C from the corresponding point on the other side of the envelope. Tape the opening and you will have a tetrahedron.



Regular Polyhedra with Doweling Rods

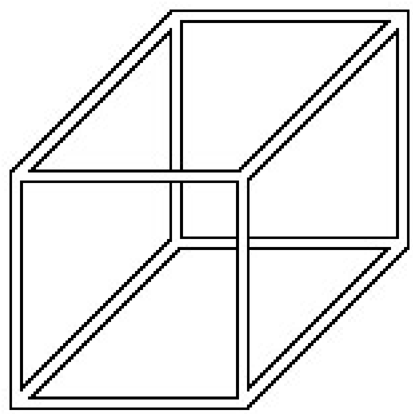
Activity 6

Four Tetrahedra
Work in four teams of two or three people. Build a tetrahedron per team (four triangular faces, three edges meeting at each vertex). One or two persons can easily build the tetrahedron. You will need 6 rods and 4 connectors. Notice that the tetrahedron is pretty stable. Explore the rotations of the tetrahedron. Rotate 120˚ around an axis that goes from the top vertex to the center of the opposite face. Rotate the tetrahedron 180˚ by holding two opposite edges by their midpoints.



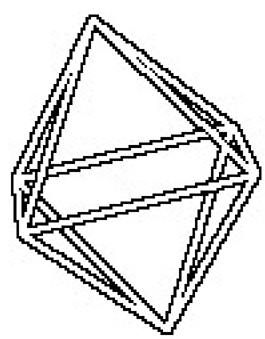
The Cube
Work in one team with three or four people. Build a cube (six square faces, three edges meeting at each vertex). It is better if at least two people participate in building the cube. You will need 12 rods, and 8 connectors. Notice that the cube is not stable at all. It can even collapse flat. When it lies flat on the floor it resembles some of the cubes drawn in textbooks.

Discuss why the cube is not stable.
In order to make it stable (keeping flexible joints) is to triangulate.
Discuss where triangulating is used in buildings, bridges, power towers, etc.



The Octahedron
Work in one team with three or four people. Build an octahedron (eight triangular faces, four edges meeting at each vertex). You will need 12 rods, and 6 connectors. Notice that the octahedron is very stable also.

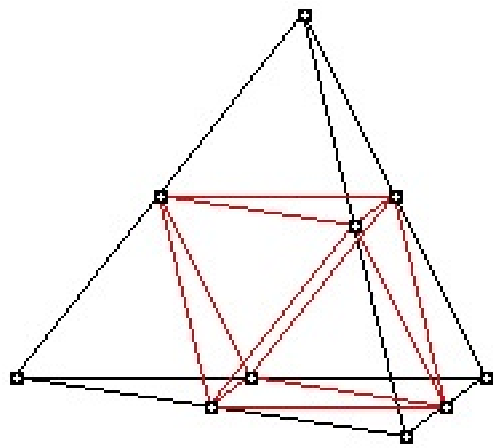
Explore the rotations of the octahedron.
Rotate 90˚ around an axis that goes from one vertex to its opposite vertex.



Regular Polyhedra with Doweling Rods

The Big Tetrahedron

You will need 36 rods and 22 connectors. The teams that built four tetrahedra and one octahedron come together. Put the octahedron with one of its faces on the ground. Put three tetrahedra around the octahedron, sharing faces. Put the fourth tetrahedron on top of the octahedron. The four tetrahedra with the octahedron in the middle form a new tetrahedron. The length of the edges of this tetrahedron is twice as big as the edges of the original tetrahedra.



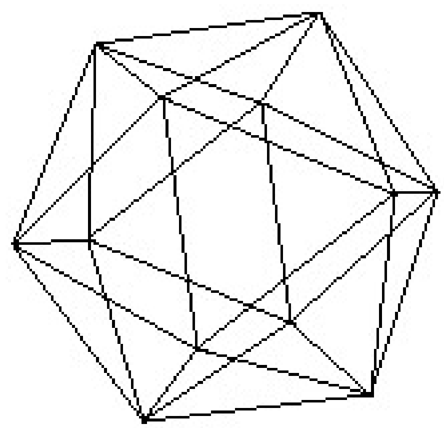
- Compare the area of the base of the big tetrahedron to the area of the base of one of the original tetrahedra. You will see that the base of the big tetrahedron is formed by four triangles. The area of the base of the big tetrahedron is therefore twice as big as the area of the base of the original tetrahedron.
- Compare the height of the big tetrahedron with the height of the original tetrahedron. The height is twice as big.
- Compare the volume of the big tetrahedron with the volume of the original tetrahedron. The volume of a tetrahedron can be computed by using the formula $v = \frac{B \times h}{3}$.

  Because the base of the tetrahedron is four times bigger, and the height is two times bigger, the volume will be 8 times bigger, $\frac{4B \times 2h}{3} = 8\frac{B \times h}{3}$.
- With this information we can figure out what is the volume of the octahedron in the middle. The volume of this octahedron will be the difference of the volume of the big tetrahedron and four times the volume of the original tetrahedron. Therefore the volume of the octahedron is four times the volume of the tetrahedron with the same edge length.

Regular Polyhedra with Doweling Rods

The Big Icosahedron

Work in a team of 5 to 8 people. Build icosahedron (20 triangular faces, 5 edges meeting at each vertex). You will need 30 rods and 12 connectors. A small-scale icosahedron may be helpful to build the big one.



Notice that the icosahedron, once completed is quite stable. It will not bend. It is not deformed. It can be made to rotate fairly easily on one of its vertices. The axis will be the line connecting the vertex on the floor with the opposite vertex.

Reference

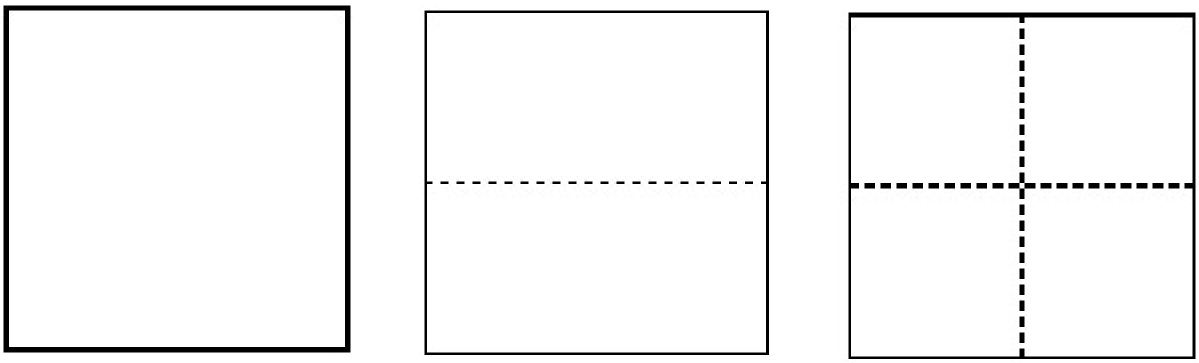
Lovitt, Charles and Clarke, Doug. *Activity Bank Volume 1*. Canberra, Australia: Curriculum Development Centre, 1988.

Photo credits

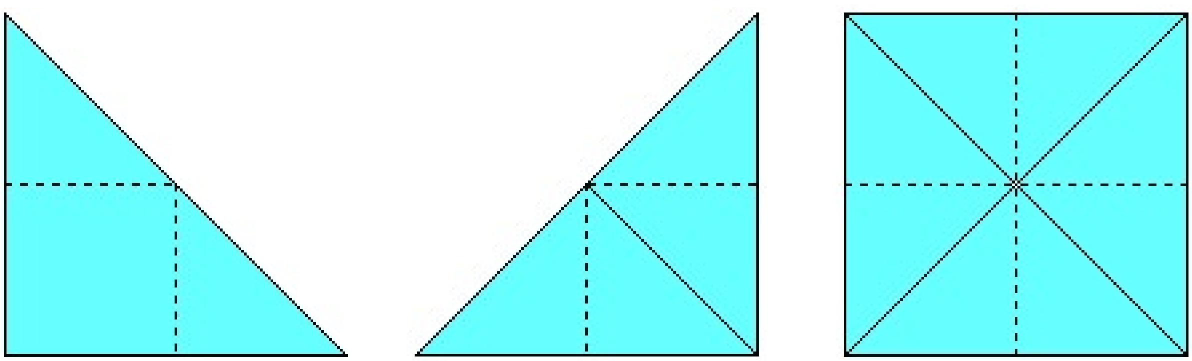
Origami Octahedron

You can find the instructions to fold the octahedron at the web site developed by Aisling Leavy
http://www.public.asu.edu/~aaafp/octahedron/octahedron.html
The photographs shown here were taken from that site.

Fold the square in half along a mid parallel (white side outside). Open again and fold in half along the other mid parallel. Open the square.



Now fold along the diagonal, color side outside. Open the square. Now fold along the other diagonal. Open the square. It will show "mountain" creases along the diagonals and "valley" creases along the mid parallels.



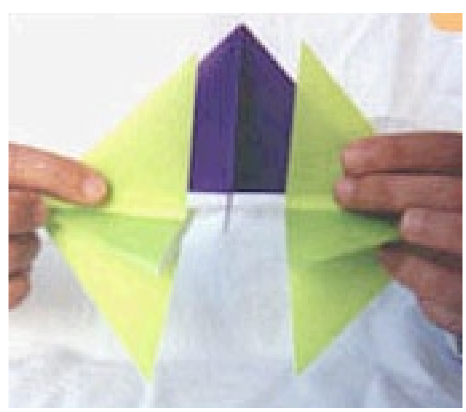
Origami Octahedron

The photographs shown here were taken from:
http://www.public.asu.edu/~aaafp/octahedron/octahedron.html

1) Form a four-pointed star.

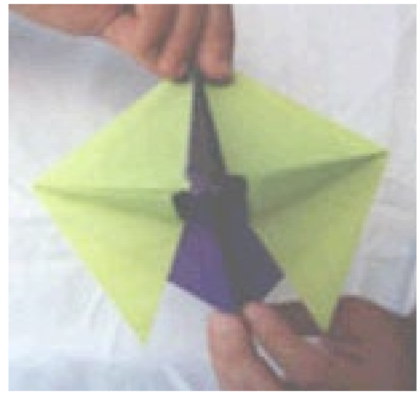
2) Repeat the process to have six stars.

3) With the six stars you can form an octahedron. Here is how. Take three units, two of the same color. Place opposite flaps of the third unit inside the pockets of the units of the same color.

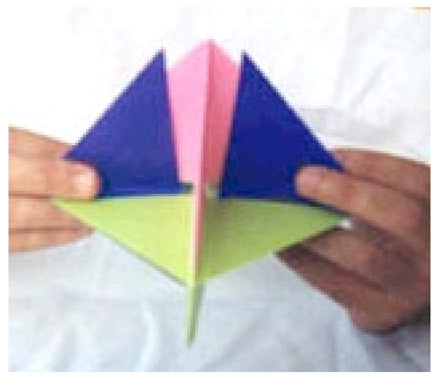


Origami Octahedron

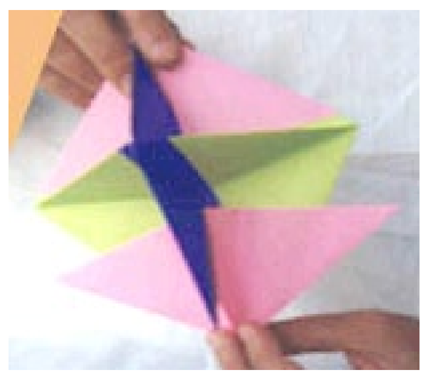
4) Take a fourth unit of the same color as the unit that was different from the other two. Place opposite flaps of this fourth unit inside the lower pockets of the other two units.



5) Rotate the octahedron. Choose a fifth unit of a different color. Insert two flaps inside, and two flaps outside.



6) Use the last unit, insert two flaps inside the pockets and have two flaps outside.



7) Each unit will have two flaps on the outside and two flaps inside another unit. Now you have the completed octahedron.