SESSION EIGHT
POLYHEDRA

Outcomes

- Know the five regular polyhedra
- Count faces, vertices, and edges of regular polyhedra
- Discover Euler's formula relating number of edges to number of vertices and faces
- Discuss concept of duality for regular polyhedra
- Determine whether a cube can be constructed with a given net
- Build a regular octahedron with a dollar bill
- Build a regular polyhedra with doweling rods
- Build a regular octahedron with origami paper

Overview

The main purpose of this session is to give participants the opportunity to construct regular polyhedra to understand different concepts related to them.

Time

10 minutes	Show segment of video *The Platonic Solids*. Give examples and non-examples of regular polyhedra.
5 minutes	Participants build cube with Polydron squares. They count faces, vertices, and edges of cube.
10 minutes	Participants build tetrahedron with Polydron triangles. Count faces, vertices, and edges.
10 minutes	Small groups build octahedron with Polydron triangles. Count faces, vertices, and edges.
10 minutes	Small groups build dodecahedron with Polydron pentagons. Count faces, vertices, and edges.
10 minutes	Small groups build icosahedron with Polydron triangles. Count faces, vertices, and edges.
10-20 minutes	Participants fill table for number of faces, vertices, and edges for all regular polyhedra. Look for patterns. Euler's formula and duality.
10-15 minutes	Explore what nets serve to build a cube
10-15 minutes	Dollar bill tetrahedron.
15 minutes	Work in teams and use doweling rods to build 4 tetrahedra (2 teams), one cube (one team), one otahedron. Explore relations and rotations of these polyhedra.
20 minutes	All participants build big icosahedron with doweling rods.
15 minutes	Work in teams of 3 participants to build octahedron with origami paper.

Materials

Facilitator	Transparencies (Eng. & Spanish)
• Video *The Platonic Solids* • Video player and screen • Examples of regular polyhedra (box, book, etc) • Non-examples of regular polyhedra (ball, jar, etc)	*No transparencies*

Materials

Participant	Handouts (English & Spanish)
• Classroom Polydron set (each team needs 12 pentagons, 6 squares, 20 triangles for icosahedron, 4 triangels for tetrahedron, 8 triangles for octahedron) • Polydron frames (the last face is hard to place if all pieces are Polydron, have a few frames for each polyhedron) • Patty paper • One crisp dollar bill • See Regular Polyhedra with Doweling rods for the following: 78 doweling rods (24 for 4 tetrahedra, 12 for cube, 12 for octahedron, 30 for icosahedron) 42 connectors of plastic tubing, screws, and bolts (16 for 4 tetrahedra, 8 for cube, 6 for octahedron, and 12 for icosahedron) • Origami paper, 6 square sheets, 2 each of 3 different colors for each group of 3 participants • Scissors • Tape • Protractor or triangle with 60° angle	**One per participant for class** *BLM 64.1-2: Regular Polyhedra Definition* *BLM 65.1-2: Building Regular Polyhedra* *BLM 66.1-6: Nets for Building Polyhedra* (optional) *BLM 67.1-4: Only Five Regular Polyhedra* *BLM 68: Nets for the Cube* *BLM 33.3: Paper Folding Geometry* (from session 4) *BLM 69.1-2: The Dollar Bill Tetrahedron* *BLM 70: Tetrahedron from Envelope* *BLM 71.1-3: Regular Polyhedra with Doweling Rods* *BLM 72.1-3: Origami Octahedron*

Activities

Preparation of Classroom	Notes
1. Place the name cards from last class near the front of the room where participants can easily find them. 2. Have participant materials and handouts on the tables. 3. Read Regular Polyhedra with Doweling Rods on page 99 to prepare for this activity before class begins.	
Platonic Solids (10 minutes)	
Materials: • Video: *The Platonic Solids* **Opening Activity** 1. Show selected segments of the video *The Platonic Solids*. The video can be used to help participants understand what regular polyhedra are. 2. Show segment corresponding to examples and non-examples of regular polyhedra. Later you may show other parts of the video, such as the explanation of why there cannot be more than five regular polyhedra.	Show only brief segments of the video. In general mathematical videos have too much information and ideas. It is better to show the segment more than once, rather than trying to cover too much of the tape.

Activities

Regular Polyhedra (45 minutes)	Notes
Materials and handouts: • Set of regular polyhedra, examples and non-examples of regular polyhedra. • BLMs 64.1-2: Regular Polyhedra Definition • BLMs 65.1-2: Building Regular Polyhedra • BLMs 66.1-6: Nets for Building Polyhedra (optional) **Description and Definition of Regular Polyhedra** 1. A set of regular polyhedra should be available for participants to handle and see. In the case of polyhedra the terminology will be a novelty for most participants. The meaning of words such as edges, vertices, and faces should be illustrated carefully on a polyhedron. The faces of a regular polyhedron are regular polygons. However, the terminology for the borders of the polygons is somewhat different depending on the context. In the case of polygons, when considered on their own, we speak of the sides of a polygon. In the context of polyhedra, we talk of edges instead. 2. Examples and non-examples of regular polyhedra should be provided, and have participant explain what makes a particular solid a regular polyhedron or not. **Building Regular Polyhedra (5 minutes)** 1. It is very instructive for participants to build their own regular polyhedra. Commercially available products such as Polydron make the construction really easy. In case of using Polydron materials, make sure you provide at least one frame face. It works as a window to see the inside of the polyhedron, and also facilitates putting the polyhedra together. 2. Usually, it is enough to direct participants as to how to put together the faces around the first vertex of the regular polyhedron. Sometimes participants get carried away and connect too many faces together, or connect them in irregular ways. Remind them that all the other vertices should look alike. In some cases participants will ask to see a completed model. For the most part, participants enjoy working with these materials. **Activity 1: The Cube** Give each participant or pair of participants six Polydron squares. Ask participants to build the cube first. They are familiar with this shape. Participants need to be aware that Polydron pieces have front and back. If two pieces do not	If no *Polydron* materials are available, participants can use the handout *Nets for Building Polyhedra* provided at the end of the activities. Ask participants to cut the nets in advance so that not too much time is spent building the polyhedra. Pasting the nets on light cardboard (such as folders) before cutting them will make sturdier models. When cardboard is used, ask participants to trace the edges that are going to be folded with a ruler and a roller pen or a pencil. That way it will be easier to fold them.

Activities

Regular Polyhedra (continued)	Notes
match, ask them to turn one of them. In order to complete the cube, all pieces should have the same orientation (for example with the trademark inside). **Activity 2: The Tetrahedron (10 minutes)** Give each participant four Polydron triangles. Ask participants to build a tetrahedron. **Activity 3: The Octahedron (10 minutes)** Give eight triangles to each small group. Ask them to build the octahedron. **Activity 4: The Dodecahedron (10 minutes)** Give 12 pentagons to each small group. Ask them to build the dodecahedron. **Activity 5: The Icosahedron (10 minutes)** Give 20 triangles to each small group. Ask them to build the icosahedron.	If they build the polyhedron cooperatively, each building a part, they need to make sure the orientation of the pieces is the same, so that they can put parts together.
Only Five Regular Polyhedra (10-20 minutes)	
Materials and handouts: • Video: *The Platonic Solids* • BLMs 67.1-4: Only Five Regular Polyhedra *The Platonic Solids* video provides a very nice proof of why there are no more than five regular polyhedra. Stop the tape to allow participants time to discuss each segment and if necessary show the segment again. **Activity 1. Counting the Faces, Edges, and Vertices of Regular Polyhedra** 1. Once participants have built their own polyhedra they should count the faces using their own method. Some will mark the faces they have already counted. Some will describe sets of faces, for example for the octahedron say "four faces on top and four faces below". 2. Participants should also count the number of vertices of their polyhedra. These activities will help them see the polyhedra from different perspectives as they turn and rotate them to make sure they count all the vertices. Again, they may count sets of vertices, for example for the cube say "four vertices on top and four vertices on the bottom." 3. Counting edges may be a little more challenging for participants. Some may find more efficient ways to count them, such as the ones described in the handout. **Activity 2. Euler's Formula for Polyhedra** 1. Once participants have counted the number of faces, edges, and vertices of polyhedra, they can write their results systematically on the table provided in their	

Activities

Only Five Regular Polyhedra (continued)	Notes
handouts. Participants should look for patterns. Some may note that for each polyhedron, the number of faces plus the number of vertices is two more than the number of edges. 2. Let F represent the number of faces, V the number of vertices, and E the number of edges. We can represent the relationship as F + V = E + 2.of vertices, and E the number of edges. We can represent the relationship as F + V = E + 2. **Activity 3. Duality** Some may note that for the cube and octahedron, the same numbers appear, but "switched". The number of faces of the cube is the same as the number of vertices of the octahedron, and conversely, the number of vertices of the cube is the same as the number of faces of the octahedron. The same relationship exists between the icosahedron and the dodecahedron. With the help of the drawings provided, or with transparent polyhedra, participants can imagine how one polyhedron fits inside its dual, so that its vertices touch the centers of the faces of the dual polyhedron.	
Nets for the Cube (10-15 minutes)	
Materials and handouts: • Paper • BLM 68: Nets for the Cube A good exercise in spatial visualization is to try to decide whether a given net will form a cube when folded. Some participants may benefit from actually folding a few of the nets to convince them whether or not a cube can be formed with the given net.	
The Dollar Bill Tetrahedron (10-15 minutes)	
Materials and handouts: • Dollar bills • BLM 33.3: Paper Folding Geometry (from session 4) • BLMs 69.1-2: The Dollar Bill Tetrahedron **A 60˚ Angle** Before doing the dollar bill activity, do the activity of folding a 60˚ angle. The handout is included in session 4 as part of the paper folding geometry section. **The Dollar Bill Tetrahedron** 1. The instructor should do the activity himself or herself before conducting the activity with participants. A crisp new dollar bill works best.	

Activities

The Dollar Bill Tetrahedron (continued)	Notes
2. Ask participants to fold a crisp dollar bill in half along the long mid parallel, and open the bill again. Ask them to fold one corner of the dollar bill onto the mid parallel, so that the crease passes through the adjacent corner to form a 60° angle. 3. They will have half of an equilateral triangle folded. Ask them to use this as a guide to fold the bill to form another 60° angle as indicated in the handout. Ask them to fold again to form another equilateral triangle. Let them fold the remaining flap. 4. When they open the dollar they should see the zigzag pattern shown in the handout. 5. For the last step, when all creases have been done, ask them to hold the two small flaps on each side together and the tetrahedron will naturally form.	For the dollar bill tetrahedron it is especially important to ensure that everybody makes the first fold right, and that everybody finishes one step before moving to the next. Some participants will need to see the fold being modeled. However, let the participants do their own folds, do not fold the paper for them.
6. An interesting discussion is whether any rectangular piece of paper folded this way will yield a tetrahedron. Participants will realize that for example, letter paper would not work, because the longer side is not long enough in relation to the shorter side. The fact that the ratio of the sides in the dollar bill is very close the ratio needed to obtain a tetrahedron is a nice coincidence.	For participants with a stronger mathematics background you can ask them to find the ratio of the two sides of a rectangle that folds exactly into a tetrahedron (the way the dollar bill was folded). They can also compare how close is the dollar bill to that shape.
Tetrahedron from Envelope (5-10 minutes)	
Materials and handouts: • Envelopes • BLM 70: Tetrahedron from Envelope This is a supplimentary activity for groups that move faster. 1. Give each participant an envelope and ask him or her to seal it. Ask them to trace on one side of the closed envelope an equilateral triangle with its side equal to the shorter side of the envelope. 2. Ask them to trace a parallel line to the shorter side through the vertex of the equilateral triangle. Ask them to cut along the line. 3. Ask them to fold along the sides of the triangle. Instruct them to push A towards B, and separate C from the corresponding point on the other side of the envelope. Ask them to tape the opening and they will have a tetrahedron.	

Activities

Tetrahedron from Envelope (continued)	Notes
4. To make it easier to pop out the tetrahedron, ask them to press the pencil or ballpoint firmly to trace the triangle, and trace an equilateral triangle on the other side of the envelope too. The reason a tetrahedron is obtained is because four equilateral triangles are formed.	

Regular Polyhedra with Doweling Rods (35 minutes)

Before class begins:
Build the connectors with flexible plastic tubing, perforated with a sharp pointed object, and joined in threes with nuts and bolts (see pictures under notes). Doweling rods are available at Home Depot and similar places (see supply list under notes).

Materials and handouts:
Sets of materials need to be prepared in advance.

- For each tetrahedron: 6 rods and 4 connectors (for the big tetrahedra 4 tetrahedra are needed)
- The cube: 12 rods and 8 connectors (If the cube is to be triangulated so that it becomes stable longer rods are needed for the diagonals (1.41 times longer). To make the cube stable 6 diagonal rods are needed .)
- The octahedron: 12 rods and 6 connectors
- The icosahedron: 30 rods and 12 connectors
- BLMs 71.1-3: Regular Polyhedra with Doweling Rods

1. This activity can be done outdoors or in a room with ample space (a learning center or a library where tables can be moved around usually has enough space).

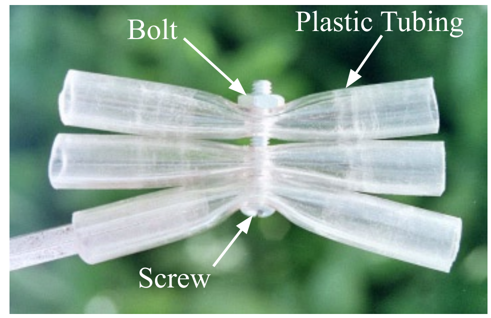
Even for participants who have constructed regular polyhedra with hands-on materials, building the polyhedra on a big scale offers many insights.

The construction of all shapes takes at least twenty minutes, so it is worthwhile to plan so that participants have the opportunity and enough time to interact with the polyhedra after they build them.

2. Once they have built the different polyhedra, invite participants to rotate them. Let them find out what angles will bring the polyhedron to a position where it looks exactly the same. This part will require guidance by the instructor, and possible modeling of how to rotate different figures with the help of some participants.

Notes:

Connector Side View



Connector Top View

To make a connector:

1. Cut plastic tubing into 2-1/2 - 3 inch sections. You will need 3 sections for each connector.
2. Perforate center of 3 stacked tube sections.
3. Insert screw into center of the 3 stacked tubes and bolt together.

(To make 42 connectors you will need to cut 126 sections of plastic tubing.)

Home Depot list:
78 wooden dowels - 48” x 3/8”
42 screws and 42 bolts
32 ft. of Plastic tubing (to fit 3/8” dowel)

Activities

Regular Polyhedra with Doweling Rods (continued)	Notes
3. Teams can build the different shapes. Depending on the size of the class they can work in stages or not. If the class is not too big, participants can form teams to build first the four tetrahedra, the cube, and the octahedron, and then all together build the icosahedron. If the class size is big, then all shapes can be built simultaneously. **Four Tetrahedra** 1. Ask participants who chose to build the tetrahedra to work in four teams of two or three people. Ask each team to build a tetrahedron. They will need six doweling rods and four joints. They will notice that the tetrahedron is pretty stable. Four tetrahedra will be needed for later. 2. Ask them to explore the rotations of the tetrahedron. Ask them to rotate it 120˚ around an axis that goes from the top vertex to the center of the opposite face. Ask them to rotate the tetrahedron 180˚ by holding two opposite edges by their midpoints. **The Cube** 1. Ask a team with three or four people to build a cube (six square faces, three edges meeting at each vertex). They will need 12 doweling rods, and 8 joiners. They will notice that the cube is not stable at all. It will collapse flat unless people hold it. Discuss why the cube is not stable. 2. Ask them to contrast the cube with other stable shapes build from the same materials, such as the octahedron and the tetrahedron. Ask them what they could do in order to make the cube stable (keeping flexible joints). **The Octahedron.** 1. Ask one team with three or four people to build an octahedron (eight triangular faces, four edges meeting at each vertex). They will need 12 rods, and six joiners. They will notice that the octahedron is very stable also. 2. Ask them to explore the rotations of the octahedron. Ask them to rotate it 90˚ around an axis that goes from one vertex to its opposite vertex. **The Big Tetrahedron** 1. Ask the teams that built four tetrahedra and one octahedron bring their polyhedra together. Let them put the octahedron with one of its faces on the ground.	One way to organize the teams to build the different polyhedra is to let small groups volunteer for each shape. Usually people choose shapes according to their level of comfort. Some prefer easier shapes, others prefer more challenging ones.

Activities

Regular Polyhedra with Doweling Rods (continued)	Notes
Ask them to put three tetrahedra around the octahedron, sharing faces. Ask them to put the fourth tetrahedron on top of the octahedron. The four tetrahedra with the octahedron in the middle form a new tetrahedron. Ask them what is the length of the edges of this tetrahedron compared to the edges of the original tetrahedra. 2. Ask them to compare the area of the base of the big tetrahedron to the area of the base of one of the original tetrahedra. They will see that the base of the big tetrahedron is formed by four triangles. The area of the base of the big tetrahedron is therefore twice as big as the area of the base of the original tetrahedron. 3. Ask them to compare the height of the big tetrahedron with the height of the original tetrahedron. The height is twice as big. 4. Ask them to compare the volume of the big tetrahedron with the volume of the original tetrahedron. Remind them that the volume of a tetrahedron can be computed by using the formula: $v = \frac{B \times h}{3}$	Participants find it surprising that using four tetrahedra and one octahedron lying on one face they can form a bigger tetrahedron. The instructor may help them by placing some of the tetrahedra in place and let the participants place the rest.
5. Participants may reason that because the base is four times bigger, and the height is two times bigger, the volume will be 8 times bigger, $\frac{4B \times 2h}{3} = 8\frac{B \times h}{3}$. 6. With this information ask them to figure out what is the volume of the octahedron in the middle. The volume of this octahedron will be the difference of the volume of the big tetrahedron and four times the volume of the original tetrahedron. Therefore the volume of the octahedron is four times the volume of the tetrahedron with the same edge length. **The Big Icosahedron** Building the icosahedron requires coordinated group work. Usually participants assign different parts of the icosahedron to a subgroup. Sometimes participants will find that they need to redo some parts or delete some rods so that the parts fit together. Having a small-scale model of the icosahedron at hand helps the team see how their parts fit together.	When compared directly, it is very hard for participants to estimate the volume of the octahedron in relation to the volume of the tetrahedron with edges of the same length. Usually the volume of the octahedron is underestimated.

Activities

Regular Polyhedra with Doweling Rods (continued)	Notes
Ask a team of 5 to 8 people to build an icosahedron (20 triangular faces, five edges meeting at each vertex). They will need 30 rods and 12 connectors. They will notice that the icosahedron, once completed is quite stable. It will not bend. It is not deformed. It can be made to rotate fairly easily on one of its vertices. The axis will be the line connecting the vertex on the floor with the opposite vertex.	Invite them to roll the big icosahedron so they can get inside and see what it looks like. Small children do this naturally; participants generally have to be encouraged to be more interactive with the shapes. A few participants will go inside the big icosahedron and see it from within. Their sense of awe is notorious.
Connections: Orgami Octahedron (15 minutes)	
Materials and handouts: • Orgami paper • BLM 72.1-3: Orgami Octahedron 1. Having three pairs of squares of different colors works best. Let participants use a complete origami paper square (6 by 6 inches) for each star. Trying smaller squares results in more frustration when assembling the final octahedron. 2. To make the orgami octahedron ask participants to follow these steps: a) Fold the square in half along a mid parallel (white side outside). b) Open again and fold in half along the other mid parallel. c) Open the square. d) Fold along the diagonal, color side outside. e) Open the square. f) Fold along the other diagonal. g) Open the square. It will show "mountain" creases along the diagonals and "valley" creases along the mid parallels. h) Form a four-pointed star as shown in the handout. i) Repeat the process to have six stars, two of each color. Tell them that they will use the six stars to form an octahedron. It works better if two people work together when assembling the six stars to form the octahedron.	Origami activities provide a good context to illustrate mathematical ideas. When conducting an origami session with a group of people it is very important they do not get lost. Make sure everybody has understood and executed one step before moving to the next. The instructor needs to walk around the classroom to make sure all participants are building the four pointed stars correctly.

Activities

Connections: Orgami Octahedron (continued)	Notes
j) Take three stars, two of the same color. Place opposite flaps of the third unit inside the pockets of the units of the same color. k) Take a fourth unit of the same color as the star that was different from the other two. Place opposite flaps of this fourth unit inside the lower pockets of the other two units. l) Use a fifth unit of a different color and insert two flaps inside, and two flaps outside. m) Use the last unit, and insert two flaps inside the pockets and have two flaps outside. 3. A crucial issue is how the final two stars are assembled. Because the last two stars go alternatively inside and outside, one person needs to hold the four stars already assembled while another person adds the fifth and sixth star. The instructor needs to make sure all participants put the pieces together correctly. The instructor may want to model how to insert the flaps inside each other at each table, or show a finished model. 4. Each star will have two flaps on the outside and two flaps inside another unit. Now you have the completed octahedron. When done in the proper way, the six stars form three interlocked squares, intersecting each other along a diagonal, which form an inner frame for the octahedron. Participants are pleasantly surprised to see those squares. 5. While the final shape will be quite stable without any gluing (as are most origami shapes), some of the last steps can be a little frustrating because the already assembled stars tend to fall apart easily when adding the last two. 6. Participants need to realize that the faces of the octahedron are not actually there. The edges of each square will form the edges of the octahedron. The faces will be indicated by three such edges.	
Closure	
Optional: Distribute Certificates of Completion	