SESSION SEVEN
SYMMETRY

Outcomes

- Identify shapes that have mirror symmetry and shapes that do not
- Identify lines of mirror symmetry for squares, equilateral triangles, regular hexagons, rhombuses
- Identify and construct shapes with point symmetry
- Construct shapes that have both point symmetry and mirror symmetry, or only one kind
- Identify and construct shapes with 90° rotational symmetry
- Establish relations between the number of shapes reflected in two hinged mirrors and the angle between the mirrors
- Realize that lines of reflection in regular polygons will give rise to the whole polygon when a hinged mirror is placed on them
- Identify angles that will give rise to a regular polygon when placed between two hinged mirrors.
- Obtain an infinite repeating pattern by using three mirrors as a kaleidoscope

Time and Overview

15-20 minutes	Participants will cut folded squares of paper to make symmetrical shapes with one or more axes of symmetry.
15–20 minutes	Participants will study the reflection of shapes with respect to an axis of mirror symmetry. They will realize that the distance to the axis of symmetry of corresponding reflected points is the same. Participants will locate the axis of symmetry in figures that are symmetrical. They will us Miras to verify that the shapes are indeed symmetrical.
10 minutes	Participants will identify all lines of symmetry in the pattern blocks.
10-15 minutes	Participants will construct with pattern blocks shapes that have mirror symmetry, point symmetry, shapes that have both kinds of symmetry, and shapes that have point symmetry but no mirror symmetry and vice versa.
20 minutes	Participants will use hinged mirrors as a simple kaleidoscope. They will use pattern blocks to explore the relation between the angle of the mirrors and the number of shapes formed by the reflections.
10 minutes	Participants will use hinged mirrors to reconstruct one pattern block by using different pairs of lines of reflection.
10 minutes	Participants will use two hinged mirrors and a third mirror to construct a kaleidoscope.

Materials

Facilitator	Transparencies (Eng. & Spanish)
• Pattern blocks for overhead projector	*BLM 55.1-3.: Patty Paper Shapes*

Materials

Participant	Handouts (English & Spanish)
• Patty paper or squares of tissue paper • Scissors • Pattern blocks (one set per table) • Mira (one per participant) • Hinged mirrors (one pair per participant) • Mirror (one per participant)	**One per participant for class** *BLM 56.1-3: Patty Paper Activities* *BLM 57.1-6: Symmetry and Asymmetry in Nature* *BLM 58.1-2: Exploring Symmetry with Pattern Blocks* **(single sided)** *BLM 59: Hinged Mirrors and Pattern Blocks* *BLM 60.1-3: Simple Kaleidoscopes* *BLM 61: Three Mirror Kaleidoscope* *BLM 62: Symmetries in the Plane* *BLM 63: Similarity*

Activities

Preparation of Classroom	Notes
1. Place the name cards from last class near the front of the room where participants can easily find them. 2. Have participant materials and handouts on the tables. 3. **BLM 62: Symmetries in the Plane** and **BLM 63: Similarity** are handouts found at the end of this session and are supplementary material for groups that move faster.	
Axes of Symmetry (15-20 minutes)	
Materials, handouts and transparencies: • Patty paper • BLMs 55.1-3: Patty Paper Shapes • BLMs 56.1-3: Patty Paper Activities **Opening Activity. Patty Paper** 1. In these activities, the important mathematical concepts are symmetric figures and axes of symmetry. The fact that symmetric points have the same distance to the axis of symmetry needs to be emphasized. 2. The use of patty paper allows to start with a square, which has two pairs of different kinds of axes of symmetry. The axes along the diagonals are one kind, the lines parallel to the borders of the square passing through the center of the square are another kind. **Activity 1. One Axis of Symmetry** 1. Instruct participants to fold a square in half to form a mid parallel. With the square folded, ask them to use the scissors to cut triangles or other shapes along the fold and along the borders. Let them predict what the paper will look like when they open it.	The first activity can also be conducted with rectangular paper. In the case of the rectangle there will be only two axes of symmetry. If the rectangle is not a square the diagonals are not axes of symmetry.

Activities

Axes of Symmetry (continued)	Notes
Activity 1 (continued) 2. Emphasize the axis of symmetry parallel to two sides of the square. Make participants aware of how corresponding parts overlap with each other when the paper is folded, and how this relates to the distance of corresponding points to the axis of symmetry. They need to realize that in symmetrical figures the distance of corresponding point to the axis of symmetry is the same.	
Activity 2. Two Axes of Symmetry 1. Ask participants to fold a new square in half to form a mid parallel, and in half again so that the second fold is perpendicular to the first. With the square folded in four parts, ask them to use the scissors to cut triangles or other shapes along the folded creases or along the borders. Ask them to predict what the paper will look like when they open it.	In this case when they open up the square they will obtain a figure that has a rotational symmetry of 180˚.
2. Emphasize the two axes of symmetry. Again, make participants aware how the distance of symmetric points to the corresponding axis is the same. Because the paper has two axes of symmetry there will be another pair of points with the same relation across the second axis.	When participants make exactly the same cuts along the two axes. In this case, the figure will have rotational symmetry of 90˚. In general, if the cuts along the two axes are not exactly the same in shape and position, the opened up square will not have 90˚ rotational symmetry.
3. The square also has also rotational symmetry of 90˚. It is interesting to observe in what cases, by cutting out figures along two axes we obtain figures that have also rotational symmetry. Some participants will make the same type of cuts along two different axes, creating thus additional kinds of symmetry that sometimes they do not anticipate.	
Activity 3. One Diagonal Axis of Symmetry 1. Ask participants to fold a new square in half, this time along one diagonal. With the square folded, ask them to use scissors to cut little triangles or other shapes along the fold and the borders.	
2. The axis of symmetry along the diagonal can make it harder for some participants to predict what the figure will look like when opened. Some people need to turn the square so that the axis of symmetry is vertical, and that way it is easier for them to see the symmetry relations.	

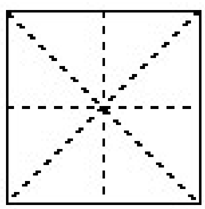
Activities

<table>
<tr><th>Axes of Symmetry (continued)</th><th>Notes</th></tr>
<tr><td>

Activity 4. Two Diagonal Axes of Symmetry
Ask participants to fold a new square in four along the two diagonals. With the square folded, ask them to use scissors to cut triangles along the folds or along the borders. Predict what the paper will look like when you open it.

Again in this case it is easier for some people to see the axes of symmetry if the square is rotated.

Activity 5. Four Axes of Symmetry
1. Ask participants to fold a square in four by making a mid parallel and a second perpendicular folds. Now ask them to fold the four-sheet square again along the diagonal that goes through the center of the original square. That is, when they open the folded square, the creases should look like the figure below:

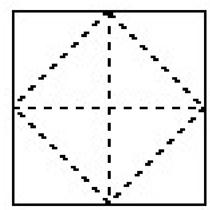


2. With the square folded, ask them to use scissors to cut triangles along the folds or along the borders. Predict what the paper will look like when you open it.

3. When the paper is folded along the two central axes and along one diagonal, the resulting figures will be symmetrical along four axes of symmetry. Make sure participants see how the figure is symmetrical with respect to both diagonals, and with respect to the axes parallel to two sides of the border. In this case, the resulting figure will also have 90° rotational symmetry.

</td><td>

Occasionally, some participants will fold the four-sheet square along the wrong diagonal. When they open the square they will obtain creases like the figure below.



When this happens, the symmetry of the figure will be different.

</td></tr>
<tr><th>Symmetry and Asymmetry in Nature (15-20 minutes)</th><th></th></tr>
<tr><td>

Materials and handouts:

- Miras
- BLMs 57.1-6: Symmetry and Asymmetry in Nature

Activity 1. Asymmetrical Objects
Ask participants to give examples in nature or in their environments of objects that are symmetrical and objects that are not symmetrical. In the beginning it is likely that all examples will involve mirror symmetry. Later activities will expand this initial concept of symmetry to include other kinds of symmetry.

</td><td></td></tr>
</table>

Activities

Symmetry and Asymmetry in Nature (continued)	Notes
Activity 2. Mirror Symmetry Participants are quite familiar with mirrors, but it is important to make the connection to the previous examples, discuss symmetrical and asymmetrical objects, and make explicit that the distance between a point and the axis of symmetry is the same as the distance from axis to the reflected point. Ask participants to use transparent mirrors such as *Mira* to verify whether a two dimensional object or drawing has mirror symmetry or not. **Activity 3. Point Symmetry** Participants can connect some additional point and their reflected images across the fixed point and verify that indeed the distance is the same. Having a copy of the object on a transparency provides a good way to see that point symmetry is also equivalent to an 180˚ rotation. **Activity 4. Rotations** The instructor can guide participants to see how the figure that has a certain kind of symmetry looks the same after a transformation that preserves that symmetry. For example, a square, after is has been rotated 90˚ looks the same as before. However, if it is only rotated 45˚ it will look different. **Activity 5. Translations (optional)** 1. Friezes or other repeating patterns are examples of objects that remain invariant under translations. Of course participants will need to imagine that the pattern extends indefinitely in both directions. 2. In the beginning it is difficult for participants to distinguish between different kinds of symmetry for frieze patterns. Again, having a transparent copy of each example allows participants to rotate, flip, and translate and see whether the transformed pattern can be overlapped with the example in the handout. 3. A good activity is to do a "scavengers hunt", asking participants to identify friezes with as many kinds of symmetry as possible. Many of the books listed at the end of the activities provide good examples. The instructor needs to verify that the examples identified in the books do indeed correspond to the kinds stated by participants. Symmetries that combine rotation and reflection are sometimes hard for participants to distinguish.	To generalize the concept of symmetry, the idea of invariance with respect to a transformation is important.

Activities

Symmetry and Asymmetry in Nature (continued)	Notes

Activity 6. Two Dimensional Patterns (optional)
Many of Escher's drawings are excellent to illustrate different kinds of symmetry. The point is not to do a systematic study of all the different kinds of patterns. Rather, participants should see that the drawings are based on simple mathematical concepts, although they can be very complex from the artistic point of view.

Activity 7. Similarity (optional)
Objects that are similar have corresponding lengths proportional and corresponding angles congruent. Some fractals are objects that are self similar, that is, a dilated part looks exactly like the whole.

Axes of Symmetry of Pattern Block Shapes (20-25 minutes)	

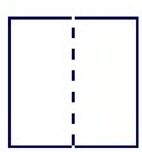
Materials and handouts:

- Miras and pattern blocks
- BLMs 58.1-2: Exploring Symmetry with Pattern Blocks

Activity 1

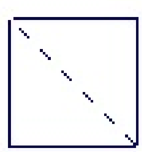
1. Ask participants to determine which of the pattern blocks have mirror symmetry. Let them use a Mira to verify that the shape is indeed symmetric. If a particular pieces has more than one kind of axis of symmetry they should note so. Let them trace the axes of symmetry.

2. Pattern blocks have different kinds of axes of symmetry. The square, for example, has two kinds of axes. One kind connects the midpoint of opposite sides.



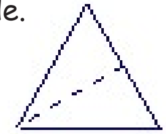
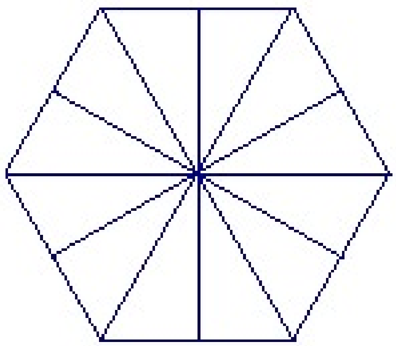
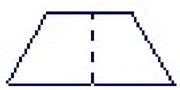
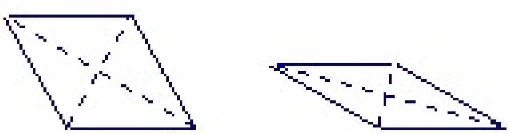
The other kind connects opposite vertices.

Notes: Sometimes participants rotate the square so that the axis of symmetry is parallel to the side of the page.

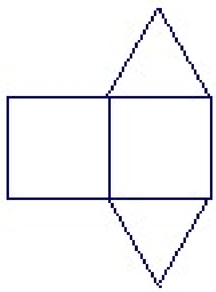
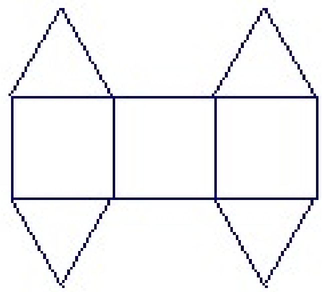
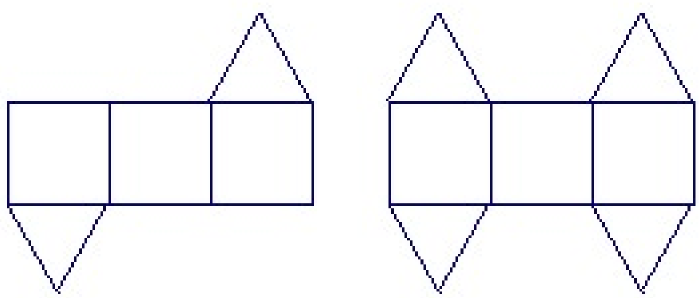


Altogether, the square has four axes of symmetry, two of each kind.

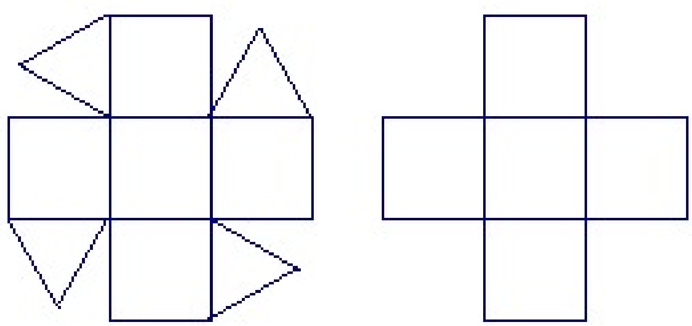
Activities

Axes of Symmetry of Pattern Block Shapes (continued)	Notes
3. The equilateral triangle has one kind of axis of symmetry. An axis of symmetry goes from one vertex to the midpoint of the opposite side. 	Often participants rotate the triangle so that one base is parallel to the border of the page and the vertex is pointing up.
4. The regular hexagon has two kinds of axes of symmetry. One kind joins opposite vertices. The other kind joins the midpoints of opposite (parallel) sides. The regular hexagon has six axes of symmetry. 	
5. The isosceles trapezoid has only one axis of mirror symmetry. It joins the midpoints of the parallel sides. 	
6. The blue rhombus and the tan rhombus have two axes of symmetry. In each rhombus, the axes are perpendicular to each other 	Sometimes participants rotate the figures to see the axes of symmetry more clearly.
Activity 2 1. Ask participants to combine several pattern blocks to design a shape that is symmetrical (mirror symmetry). Let them use a Mira if necessary to verify that their design is indeed symmetrical. 2. Participants will create symmetrical shapes, some very simple, some quite elaborate. Some shapes will have one axis of symmetry only, other shapes will have more than one axis of symmetry.	Most participants do not have a problem creating shapes that have mirror symmetry. Walk around the tables to make sure participants can identify the axis of symmetry.

Activities

Axes of Symmetry of Pattern Block Shapes (continued)	Notes
Activity 2 (continued)  One axis of symmetry  Two axis of symmetry	
Activity 3 Ask participants to construct a design that has point symmetry. Let them identify the center of symmetry. They may trace their design on paper, and trace lines across the point of symmetry to identify points on their design that are symmetrical with respect to the center of symmetry.  Shapes with point symmetry	The instructor can share on the overhead projectors some of the designs created by participants that have point symmetry. Some participants will need help identifying the center of symmetry.
Activity 4 Ask participants to construct a design with pattern blocks that has point symmetry but does not have mirror symmetry.	The first figure in activity 3 satisfies this condition.
Activity 5 Construct a design with pattern blocks that has both point symmetry and mirror symmetry.	The second figure in activity 3 satisfies this condition.
Activity 6 Ask participants to verify by rotating that a shape with point symmetry also has a rotational symmetry of 180° around the center of symmetry.	Some participants may have difficulty with this activity. Tracing one copy of the shape on translucent paper may help them do the rotation to see better.

Activities

Axes of Symmetry of Pattern Block Shapes (continued)	Notes
Activity 7 (optional) Ask participants to verify that the following two conditions are equivalent using several shapes. That is, a shape that satisfies condition a) also satisfies condition b) and vice versa. a) A shape has point symmetry and one axis of mirror symmetry. b) A shape has two perpendicular axes of mirror symmetry.	
Activity 8 Ask participants to identify which of the pattern block pieces have point symmetry.	All pattern block shapes, except the isosceles trapezoid have point symmetry.
Activity 9 Ask participants to verify that the square has rotational symmetry of 90° with respect to its center.	
Activity 10 Ask participants to construct a design that has rotational symmetry of 90°. Identify the center of rotation 	Participants frequently use a square as a base to construct their designs with rotational symmetry of 90˚.
Activity 11 Ask participants to construct a design with pattern blocks that has rotational symmetry of 90° but does not have mirror symmetry.	The first figure in activity 10 satisfies this condition
Activity 12 Construct a design with pattern blocks that has both rotational symmetry of 90° and has also mirror symmetry. Identify all the symmetry axes of your design.	
Activity 13 Ask participants to identify the rotational symmetries of the equilateral triangle.	The equilateral triangle can be rotated by multiples of 120˚ and appear unchanged.

Activities

Axes of Symmetry of Pattern Block Shapes (continued)	Notes
Activity 14 Ask participants to identify the rotational symmetries of the regular hexagon.	The regular hexagon can be rotated by multiples of 60˚ and appear unchanged.
Activity 15 Ask participants whether there are any other pattern block pieces that have rotational symmetry of any kind.	The two rhombuses have rotational symmetry of 180˚.

Hinged Mirrors and Pattern Blocks (20 minutes)	
Materials and handouts: • Hinged mirrors and pattern blocks • BLM 59: Hinged Mirrors and Pattern Blocks	
1. Have participants place a square between the two hinged mirrors so that they form a 90˚ angle. Ask: *"How many squares do you see?"* (Including the original.)	Participants should be able to see a total of four squares.
2. Have participants place an equilateral triangle between the two hinged mirrors so that they form a 60˚ angle. Ask: *"How many triangles do you see?"*	They should see a total of six equilateral triangles. They will also notice that they form a hexagon.
3. Have participants place a hexagon between the two hinged mirrors so that they form a 120˚ angle. Ask: *"How many hexagons do you see?"*	They will see a total of three hexagons.
4. Have participants place the tan rhombus between the two mirrors so that they form a 30˚ angle. Ask: *"How many rhombuses do they see?"*	They should be able to see twelve rhombuses.
5. Have participants fill the table in the handout. The completed table should show these values.	

Shape	Angle Between Mirrors	Number of Shapes
Tan rhombus	30˚	12
Triangle	60˚	6
Square	90˚	4
Hexagon	120˚	3

Ask participants:
"What do you notice about the values in the table?"

Notes: Some participants will express the relation saying that the smaller the angle the more figures there are. Others will express the relation in more precisely as the number of figures times the angle is 360.

Activities

Hinged Mirrors and Pattern Blocks (continued)	Notes
6. Have participants place the other pattern block shapes between the mirrors in different positions. For example, place the blue rhombus to form first a 60˚ angle between the mirrors, and then to form a 120˚ angle. Ask: *“Is the number of shapes consistent with the relation suggested by the table?”*	When the mirrors form an angle of 120˚ they will see three rhombuses. Here again so that 3 × 120 = 360. When the mirrors form an angle of 60˚ they will see six rhombuses forming a star.
7. Have participants express the relation between the size of the angle and the number of rhombuses with an equation.	Participants will figure out that the angle has to be 360˚ ÷ 5 = 72˚.
8. Have participants place the tan rhombus between the two mirrors and open the angle so that they see exactly five rhombuses (including the original). Ask: *“What is the angle formed by the mirrors?”*	
Connections: Simple Kaleidoscopes (10 minutes)	
Materials and handouts: • Hinged mirrors, tiny objects • BLMs 60.1-3: Simple Kaleidoscopes	
Activity 1. Hexagon 1. Have participants draw all the mirror lines of the hexagon. Let them place the hinged mirrors along two mirror lines, with the mirrors meeting at the center of the hexagon. If they look between the mirrors they should see the entire hexagon.	If they place a few tiny objects between the two mirrors, they will see a pattern that is kaleidoscopic.
2. Let participants explore to see whether they can reconstruct the whole hexagon by placing the hinged mirrors along the indicated mirror lines. Ask: *“What can you say about the angles for which you can reconstruct the entire hexagon with the hinged mirrors?”* With some prompting, participants will realize that the angles that give rise to the whole hexagon are multiples of 30˚.	With some practice, participants will be able to reconstruct a whole hexagon by placing the hinged mirror on an hexagon that has no symmetry lines marked.
Activity 2. Square Have participants draw the mirror lines in a square. Now have them place the mirrors along two mirror lines, with the mirrors meeting in the center of the square. If they look between the mirrors they should see the entire square. Ask: *“What can you say about the angles that will form a complete square when reflected?”* With some guidance, participants should realize that the angles that form the whole square again are multiples of 45˚.	With some practice, participants will be able to reconstruct a whole square by placing the hinged mirror on a square that has no symmetry lines marked. If they place a few tiny objects between the two mirrors, they will see a pattern that is kaleidoscopic.

Activities

Connections: Simple Kaleidoscopes (continued)	Notes
Activity 3 1. Have participants place mirrors along the lines that meet at the marked point. Ask: *"Which of the angles produce a regular polygon?"* 2. They may also try their own angles. Ask them to express the condition satisfied by angles that give rise to a regular polygon in their own words.	In order to produce a regular polygon, the marked angle has to divide evenly into 360˚.
Three Mirror Kaleidoscopes (10 minutes)	
Materials and handouts: • Hinged mirrors and single mirrors • BLM 61: Three Mirror Kaleidoscopes 1. The kaleidoscope made of two mirrors generates finite patterns with the symmetries of the regular polygons. There are also more complex kaleidoscopes whose patterns appear to be infinite in extent. Three mirrors joined to form a prism are used. Participants can easily use two equal mirrors that are hinged and add a third mirror of the same size to form a kaleidoscope. 2. Ask participants to use three mirrors to form a prism whose base is an equilateral triangle. Ask them to place colored patterns in the base of the kaleidoscope.	Kaleidoscopes that produce infinite patterns can also be obtained with triangles other than the equilateral. The angles for one or these kaleidoscopes are 30˚, 60˚, and 90˚. For the other the angles are 90˚, 45˚, and 45˚.
Closure	
Symmetry is a simple form to find patterns in nature. The concept of symmetry has been of great importance in the study of crystals and in elementary particles. The concept of invariance under transformation can also be extended to other type of transformations (in non-Euclidean geometries) The use of objects that are self-similar has a great potential in the study of phenomena such as the study of chaos with the help of fractal geometry.	
Take Home Activities	
Take home and complete optional activities and all other activities not finished in class.	
Preparation for the Next Session	
Collect name cards for use in the next sessions.	